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Local antimagic vertex total coloring on fan graph and graph resulting from comb product operation

R Amalia[1] and Masruroh[1]

[1]Mathematics Department, Mathematics and Science Faculty, Universitas Islam Madura Pamekasan, Indonesia

E-mail: rica.amalia@uim.ac.id

Abstract. Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$. A bijection $f: V(G) \cup E(G) \rightarrow \{1,2,3, \dots, n + m\}$ is called local antimagic vertex total coloring if for any two adjacent vertices u and v, $w_t(u) \neq w_t(v)$, where $w_t(u) = \sum_{e \in E(u)} f(e) + f(u)$, and $E(u)$ is a set of edges incident to u. Thus any local antimagic vertex total labeling induces a proper vertex coloring of G where the vertex v is assigned the color $w_t(v)$. The local antimagic vertex total chromatic number $\chi_{lvat}(G)$ is the minimum number of colors taken over all colorings induced by local antimagic vertex total. In this paper we investigate local antimagic vertex total coloring on fan graph (F_n) and graph resulting from comb product operation of F_n and F_3 which denoted by $F_n \rhd F_3$. We get two theorems related to the local antimagic vertex total chromatic number. First, $\chi_{lvat}(F_n) = 3$ where $n \geq 3$. Second, $3 \leq \chi_{lvat}(F_n \rhd F_3) \leq 5$ where $n \geq 3$.

1. Introduction

There are many topics in graph theory, one of them is coloring. In general, graph coloring is the giving of color to elements on the graph so that neighboring elements have a different color. Based on its element, graph coloring is divided into three kinds, namely vertex coloring, edge coloring and regional coloring. The detail of this theory can be read in [7] and [8].

The simple concept of graph coloring then has been developed into graph labeling. In general, graph labeling is the giving of labels, which is natural numbers, to the elements on the graph such as vertices or edges, or both [12]. Graph labeling is divided into two, namely magic labeling and antimagic labeling. Magic labeling on graph $\boldsymbol{G}$ is labeling the elements on $\boldsymbol{G}$ such that the sum of the labels of all elements incident with any vertex is the same [9]. On the contrary, if the sum of the labels of all elements incident with any two neighboring vertex is different its called antimagic labeling.

The research on antimagic labeling continues to be carried out by many researches. Dafik, *et al.* built the concept of super edge-antimagic total labeling [5, 6]. Arumugam, *et al.* have the concept of local antimagic vertex coloring [4]. Agustin, *et al.* got the concept of local edge antimagic coloring [1] which then developed to super local edge antimagic total coloring [2, 3, 10, 11]. Next, Putri, *et al.* built the concept of local vertex antimagic total coloring [13] which then continued by Kurniawati, *et al.* [12].

The concept of local vertex antimagic total coloring is explained as follows. Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be a connected graxph with $|\boldsymbol{V}| = \boldsymbol{n}$ and $|\boldsymbol{E}| = \boldsymbol{m}$. A bijection $\boldsymbol{f}: \boldsymbol{V}(\boldsymbol{G}) \cup \boldsymbol{E}\,(\boldsymbol{G}) \rightarrow \{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \dots, \boldsymbol{n} + \boldsymbol{m}\}$ is called local antimagic vertex total coloring if for any two adjacent vertices $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{w_t(u) \neq w_t(v)}$, where $\boldsymbol{w_t(u) = \sum_{e \in E(u)} f(e) + f(u)}$, and $\boldsymbol{E(u)}$ is a set of edges incident to $\boldsymbol{u}$. Thus any local antimagic vertex total labeling induces a proper vertex coloring of $\boldsymbol{G}$ where the vertex $\boldsymbol{v}$ is assigned the color $\boldsymbol{w_t(v)}$. The local antimagic vertex total chromatic number $\boldsymbol{\chi_{lvat}(G)}$ is the minimum number of colors taken over all colorings induced by local vertex antimagic total labelings of $\boldsymbol{G}$ [12].

In the previous research, we got the local antimagic vertex total coloring on some families tree [13] and graphs with homogeneous pendant vertex [12]. Based on this, the author conduct further research on local antimagic vertex total coloring on fan graph $(\boldsymbol{F_n})$ and graph resulting from comb product operation, namely $\boldsymbol{F_n \rhd F_3}$.

Lemma 1.1 [13] If $\chi(G)$ is a vertex coloring chromatic number of graph G then $\chi_{lvat}(G) \geq \chi(G)$

2. Local Antimagic Vertex Total Coloring on Fan Graph

Before we got the local antimagic vertex total coloring of fan graph, we must know its vertex coloring chromatic number.

Lemma 2.1 If F_n is fan graph then the vertex coloring chromatic number of F_n is $\chi(F_n) = 3$

Proof. Vertex coloring of graph G is coloring the vertices in G such that any two adjacent vertices in $V(G)$ have diferent color. We have vertex set $V(F_n) = \{x\} \cup \{x_i;\ 1 \leq i \leq n\}$ and edge set $E(F_n) = \{xx_i;\ 1 \leq i \leq n\} \cup \{x_i x_{i+1};\ 1 \leq i \leq n-1\}$. So, the color of vertex x must be different than the color of vertex x_i and the color of vertex x_i must be different than the color of vertex x_{i+1}. If we gave a color for vertex x, i.e: color 1, then the color of vertices x_i, for $i = 1,2,\ldots,n$, can not be color 1. Next if we gave vertex x_i color 2 then the color of vertex x_{i+1} can not be color 2, for $i = 1,2,\ldots,n-1$. So, we have 3 colors for F_n and this is the minimum color. Therefore $\chi(F_n) = 3$. ■

Theorem 2.1 For $\boldsymbol{n \geq 3}$, $\boldsymbol{n}$ natural numbers, the local antimagic vertex total chromatic number of fan graph is $\boldsymbol{\chi_{lvat}(F_n) = 3}$.

Proof. Fan graph, denoted by $\boldsymbol{F_n}$, have vertex set $\boldsymbol{V(F_n) = \{x\} \cup \{x_i;\ 1 \leq i \leq n\}}$ and edge set $\boldsymbol{E(F_n) = \{xx_i;\ 1 \leq i \leq n\} \cup \{x_i x_{i+1};\ 1 \leq i \leq n-1\}}$. The cardinality of vertex set and edge set of fan graph, respectively, is $\boldsymbol{|V(F_n)| = n+1}$ and $\boldsymbol{|E(F_n)| = 2n-1}$. To prove this theorem we have 2 cases, namely when $\boldsymbol{n}$ is odd and when $\boldsymbol{n}$ is even.

Case 1. For $\boldsymbol{n}$ odd number, to prove that $\boldsymbol{\chi_{lvat}(F_n) = 3}$, it must be proved that $\boldsymbol{\chi_{lvat}(F_n) \geq 3}$ and $\boldsymbol{\chi_{lvat}(F_n) \leq 3}$. Next, it will be proven that $\boldsymbol{\chi_{lvat}(F_n) \leq 3}$ by labeling fan graph $\boldsymbol{F_n}$ using the function of $\boldsymbol{f: V(F_n) \cup E(F_n) \to \{1, 2, 3 \ldots, 3n\}}$.

The vertex labeling functions are

$$\boldsymbol{f(x) = 3n}$$

$$\boldsymbol{f(x_i) = \begin{cases} \dfrac{5n-i-2}{2}, & \text{if } i \text{ is odd}, \quad i \neq n \\ 3n - \dfrac{i}{2}, & \text{if } i \text{ is even} \\ \dfrac{5n-1}{2}, & i = n \end{cases}}$$

The edge labeling functions are

$$\boldsymbol{f(x_i x_{i+1}) = \begin{cases} \dfrac{i+1}{2}, & \text{if } i \text{ is odd} \\ \dfrac{2n-i}{2}, & \text{if } i \text{ is even} \end{cases}}$$

$$\boldsymbol{f(xx_i) = \begin{cases} 2n-1, & i = 1 \\ \dfrac{2n+i-3}{2}, & \text{if } i \text{ is odd}, \quad i \neq 1 \\ \dfrac{3n+i-3}{2}, & \text{if } i \text{ is even} \end{cases}}$$

It will be proven that function f is local antimagic vertex total coloring by proving that f is a bijection and neighboring vertices have different weights. First it will be proven that f is a bijection. It is known that $R_f = \{1, 2, 3, ..., 2n-1\} \cup \{2n, 2n+1, 2n+2, ..., 3n-1\} \cup \{3n\}$ so range of function f is $R_f = \{1, 2, 3, ..., 3n\}$. It is clear that range and codomain have the same cardinality so f is a surjective function. Next, it will be proven that f is an injection. For any $u, v \in V(F_n)$ and $u \neq v$ applies $f(u) \neq f(v)$, so f is an injective function. Since f is surjection and injection, then f is a bijection.

By function f, we get the total vertex weight of fan graph (F_n). The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2}n(3n+5)$$

$$w_t(x_i) = \begin{cases} \frac{9n-3}{2}, & \text{if } i \text{ is odd} \\ \frac{11n-3}{2}, & \text{if } i \text{ is even} \end{cases}$$

Based on the total vertex weight function, it will be shown that neighboring vertices have different weights. Vertex x is adjacent to vertex x_i and vertex x_i is adjacent to vertex x_{i+1}, so it must have different weights. First, we assume that vertex x has the same weight as vertex x_i. It can be stated that $w_t(x) = w_t(x_i)$ or $\frac{1}{2}n(3n+5) = \frac{9n-3}{2} = \frac{11n-3}{2}$. If the equation is solved, we get the value of $n = 0$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex x has a different weight from vertex x_i. Next, we assume that vertex x_i has the same weight as vertex x_{i+1}. If i is odd then $i+1$ is even and vice versa. So we have $w_t(x_i) = w_t(x_{i+1})$ or $\frac{9n-3}{2} = \frac{11n-3}{2}$ or $n = 0$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex x_i has a different weight from vertex x_{i+1}. It can be concluded that neighboring vertices have different weights or $w_t(x) \neq w_t(x_i)$ and $w_t(x_i) \neq w_t(x_{i+1})$.

Since each neighboring vertex has different weights and f is a bijection, it can be concluded that f is local antimagic labeling. Then the vertices on fan graph (F_n) are colored according to its total weight, which it is called local antimagic vertex total coloring. Because we have 3 total weights, then we also have 3 colors. Therefore, the local antimagic vertex total chromatic number of fan graph is $\chi_{lvat}(F_n) \leq 3$.

Next it will be shown that $\chi_{lvat}(F_n) \geq 3$. Based on Lemma 1.1. and Lemma 2.1, we have $\chi_{lvat}(F_n) \geq \chi(F_n) = 3$. Then $\chi_{lvat}(F_n) \geq 3$. Because $\chi_{lvat}(F_n) \leq 3$ and $\chi_{lvat}(F_n) \geq 3$, it is proven that $\chi_{lvat}(F_n) = 3$.

As an illustration, it is presented by Figure 1 which is the local antimagic vertex total coloring of F_7.

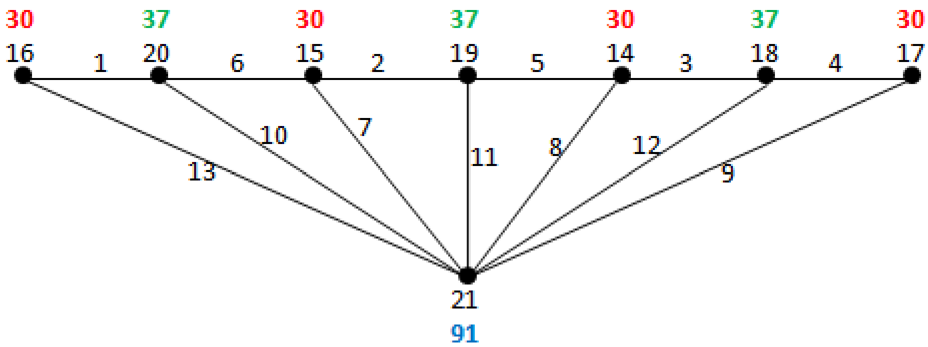

Figure 1. Illustration of local antimagic vertex total coloring on F_7.

Case 2. For $\boldsymbol{n}$ even number, to prove that $\chi_{lvat}(F_n) = 3$, it must be proved that $\chi_{lvat}(F_n) \geq 3$ and $\chi_{lvat}(F_n) \leq 3$. Next, it will be proven that $\chi_{lvat}(F_n) \leq 3$ by labeling fan graph F_n using the function of $f: V(F_n) \cup E\,(F_n) \rightarrow \{1, 2, 3\, \ldots, 3n\}$.

The vertex labeling functions are

$$f(x) = 3n$$

$$f(x_i) = \begin{cases} \dfrac{5n - i - 1}{2}, & \text{if } i \text{ is odd} \\ \dfrac{6n - i - 2}{2}, & \text{if } i \text{ is even}, \quad i \neq n \\ 3n - 1, & i = n \end{cases}$$

The edge labeling functions are

$$f(x_i x_{i+1}) = \begin{cases} \dfrac{i + 1}{2}, & \text{if } i \text{ is odd} \\ \dfrac{2n - i}{2}, & \text{if } i \text{ is even} \end{cases}$$

$$f(xx_i) = \begin{cases} 2n - 1, & i = 1 \\ \dfrac{2n + i - 3}{2}, & \text{if } i \text{ is odd} \\ \dfrac{3n + i - 4}{2}, & \text{if } i \text{ is even} \end{cases}$$

It will be proven that function f is the local antimagic vertex total coloring by proving that f is a bijection and neighboring vertices have different weights. First it will be proven that f is a bijection. It is known that $R_f = \{1, 2, 3, \ldots, 2n - 1\} \cup \{2n, 2n + 1, 2n + 2, \ldots, 3n - 1\} \cup \{3n\}$ so the range of f is $R_f = \{1, 2, 3, \ldots, 3n\}$. It is clear that range and codomain have the same cardinality so f is a surjective function. Next, it will be proven that f is an injection. For any $u, v \in V(F_n)$ and $u \neq v$ applies $f(u) \neq f(v)$, so f is an injective function. Since f is surjection and injection, then f is a bijection.

By function f we get the total vertex weight of fan graph (F_n). The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2} n(3n + 5)$$

$$w_t(x_i) = \begin{cases} \dfrac{9n - 2}{2}, & \text{if } i \text{ is odd} \\ \dfrac{11n - 6}{2}, & \text{if } i \text{ is even} \end{cases}$$

Based on total vertex weight function above, it will be shown that neighboring vertices have different weights. Vertex x is adjacent to vertex x_i and vertex x_i is adjacent to vertex x_{i+1}, so it must have different weights. First, we assume that vertex x has the same weight as vertex x_i. It can be stated that $w_t(x) = w_t(x_i)$ or $\frac{1}{2} n(3n + 5) = \frac{9n-2}{2} = \frac{11n-6}{2}$. If the equation is solved, we get the value of $n = 2$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex x has a different weight from vertex x_i. Next, we assume that vertex x_i has the same weight as vertex x_{i+1}. If i is odd then $i + 1$ is even and vice versa. So we have $w_t(x_i) = w_t(x_{i+1})$ or $\frac{9n-2}{2} = \frac{11n-6}{2}$ or $n = 2$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex x_i has a different weight from vertex x_{i+1}. It can be concluded that neighboring vertices have different weights or $w_t(x) \neq w_t(x_i)$ and $w_t(x_i) \neq w_t(x_{i+1})$.

Since each neighboring vertex has different weights and f is a bijection, it can be concluded that f is local antimagic labeling. Then the vertices on the fan graph (F_n) are colored according to its total weight, which it is called local antimagic vertex total coloring. Because we have 3 total weights, then we also have 3 colors. Therefore, the local antimagic vertex total chromatic number of fan graph is $\boldsymbol{\chi_{lvat}(F_n) \leq 3}$.

Next it will be shown that $\boldsymbol{\chi_{lvat}(F_n) \geq 3}$. Based on Lemma 1.1. and Lemma 2.1, we have $\boldsymbol{\chi_{lvat}(F_n) \geq \chi(F_n) = 3}$. Then $\boldsymbol{\chi_{lvat}(F_n) \geq 3}$. Because $\boldsymbol{\chi_{lvat}(F_n) \geq 3}$ and $\boldsymbol{\chi_{lvat}(F_n) \leq 3}$, it is proven that $\boldsymbol{\chi_{lvat}(F_n) = 3}$.

As an illustration, it is presented by Figure 2 which is the local antimagic vertex total coloring of $\boldsymbol{F_8}$.

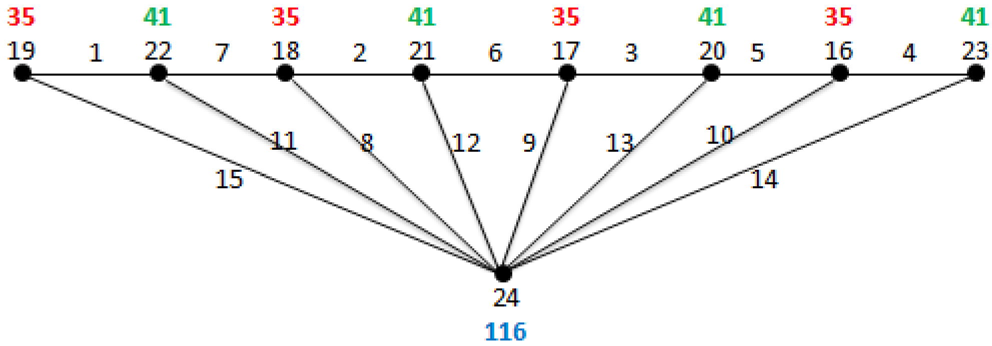

Figure 2. Illustration of local antimagic vertex total coloring on $\boldsymbol{F_8}$.

3. Local Antimagic Vertex Total Coloring on Fan Graph with Comb Product Operation

In this section, we got the local antimagic vertex total coloring of fan graph with comb product operation that is $F_n \rhd F_3$. Based on the definition of comb product operation, the obtained result of observation is as follows:

Observation 3.1 Let $F_n \rhd F_3$ be a graph resulting from comb product operation by taking one copy of F_n and $|V(F_n)|$ copies of F_3 and grafting the i-th copy of F_3 at the center vertex to i-th vertex of F_n. Then we have the vertex set of $F_n \rhd F_3$ is $V(F_n \rhd F_3) = \{x\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_i;\ 1 \leq i \leq 3\} \cup \{y_i^j;\ 1 \leq i \leq n, 1 \leq j \leq 3\}$ and edge set of $F_n \rhd F_3$ is $E(F_n \rhd F_3) = \{xx_i;\ 1 \leq i \leq n\} \cup \{x_i x_{i+1};\ 1 \leq i \leq n-1\} \cup \{x_i y_i^j;\ 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{xy_i;\ 1 \leq i \leq 3\} \cup \{y_i^j y_i^{j+1};\ 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{y_i y_{i+1};\ 1 \leq i \leq 2\}$. So, the vertex cardinality of $F_n \rhd F_3$ is $4(n+1)$ and edge set cardinality of $F_n \rhd F_3$ is $7n + 4$.

As an illustration, it is presented by Figure 3 which is the example of fan graph with comb product operation $F_5 \rhd F_3$.

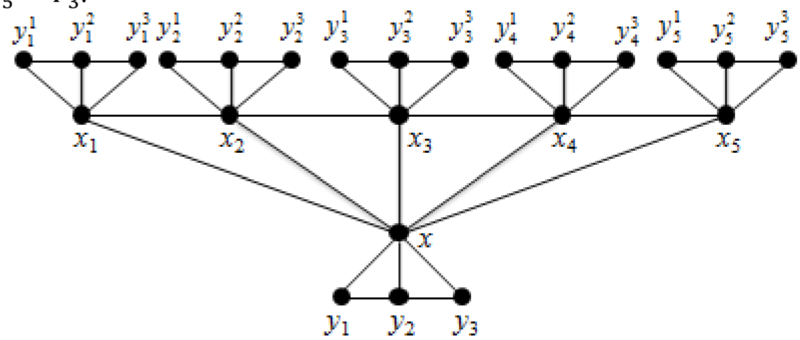

Figure 3. Illustration of fan graph with comb product operation $\boldsymbol{F_5 \rhd F_3}$.

Before we got the local antimagic vertex total coloring of fan graph with comb product operation, we must know its vertex coloring chromatic number.

Lemma 3.1 The vertex coloring chromatic number of $F_n \rhd F_3$ is $\chi(F_n \rhd F_3) = 3$

Proof. Vertex coloring of graph G is coloring the vertices in G such that any two adjacent vertices in $V(G)$ have diferent color. Based on Observation 3.1, we have vertex set $V(F_n \rhd F_3) = \{x\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_i;\ 1 \leq i \leq 3\} \cup \{y_i^j;\ 1 \leq i \leq n, 1 \leq j \leq 3\}$ and edge set $E(F_n \rhd F_3) = \{xx_i;\ 1 \leq i \leq$

$n\} \cup \{x_i x_{i+1};\ 1 \le i \le n-1\} \cup \{x_i y_i^j;\ 1 \le i \le n, 1 \le j \le 3\} \cup \{xy_i;\ 1 \le i \le 3\} \cup \{y_i^j y_i^{j+1};\ 1 \le i \le n, 1 \le j \le 2\} \cup \{y_i y_{i+1};\ 1 \le i \le 2\}$. So, the pairs of vertice that must have different color are $(x, x_i), (x_i, x_{i+1}), (x_i, y_i^j), (x, y_i), (y_i^j, y_i^{j+1})$ and $(y_i y_{i+1})$. If we gave a color for vertex x, i.e: color 1, then the color of vertices x_i, for $i = 1,2, \dots, n$, can not be color 1. Next if we gave vertex x_i color 2 then the color of vertex x_{i+1} can not be color 2, for $i = 1,2, \dots, n-1$, i.e: color 3. If vertex x has color 1 then we can give color 2 and color 3 to vertice y_i and y_{i+1} respectively. Next, if vertex x_i has color 2 then we can give color 1 and color 3 to vertice y_i^j and y_i^{j+1} respectively. And last, if vertex x_i has color 3 then we can give color 1 and color 2 to vertice y_i^j and y_i^{j+1} respectively. So, we have 3 colors for $F_n \rhd F_3$, and this is the minimum color. Therefore $\chi(F_n \rhd F_3) = 3$. ■

Theorem 3.1 Let G be a fan graph with comb product operation $F_n \rhd F_3$. For n natural numbers, $n \ge 3$, the local antimagic vertex total chromatic number of graph G is $3 \le \chi_{lvat}(G) \le 5$.

Proof. Based on Observation 3.1, G has vertex set $V(G) = \{x\} \cup \{x_i;\ 1 \le i \le n\} \cup \{y_i;\ 1 \le i \le 3\} \cup \{y_i^j;\ 1 \le i \le n, 1 \le j \le 3\}$ and edge set $E(G) = \{xx_i;\ 1 \le i \le n\} \cup \{x_i x_{i+1};\ 1 \le i \le n-1\} \cup \{x_i y_i^j;\ 1 \le i \le n, 1 \le j \le 3\} \cup \{xy_i;\ 1 \le i \le 3\} \cup \{y_i^j y_i^{j+1};\ 1 \le i \le n, 1 \le j \le 2\} \cup \{y_i y_{i+1};\ 1 \le i \le 2\}$. Vertex set and edge set cardinality of G are $|V(G)| = 4(n+1)$ and $|E(G)| = 7n+4$. To prove that $\chi_{lvat}(G) \le 5$ we have 2 cases, namely when n is odd and when n is even.

Case 1. For n odd number, it will be proven that $\chi_{lvat}(G) \le 5$ by labeling graph G using the function of $f: V(G) \cup E(G) \to \{1,2,3, \dots, 11n+8\}$.

The vertex labeling functions are

$$f(x) = 3n$$

$$f(x_i) = \begin{cases} \dfrac{4n+i-2}{2}, & \text{if } i \text{ is even} \\ \dfrac{5n+i}{2}, & \text{if } i \text{ is odd}, \quad i \ne n \\ \dfrac{5n-1}{2}, & i = n \end{cases}$$

$$f(y_i) = \begin{cases} 9n+7, & i = 1 \\ 11n+8, & i = 2 \\ 9n+11, & i = 3 \end{cases}$$

$$f(y_i^j) = \begin{cases} 11n-i+9, & j = 1, \quad i \text{ is even} \\ 10n-i+7, & i \ne n, j = 1, i \text{ is odd} \\ 10n+8, & i = n, j = 1 \\ 9n+i+6, & j = 2, \quad i \text{ is even} \\ 10n+i+8, & i \ne n, j = 2, i \text{ is odd} \\ 10n+7, & i = n, j = 2 \\ 7n+i+3, & j = 3, \quad i \text{ is even} \\ 8n+i+5, & i \ne n, j = 3, i \text{ is odd} \\ 8n+4, & i = n, j = 3 \end{cases}$$

The edge labeling functions are

$$f(x_i x_{i+1}) = \begin{cases} \dfrac{i+1}{2}, & \text{if } i \text{ is odd} \\ \dfrac{2n-i}{2}, & \text{if } i \text{ is even} \end{cases}$$

$$f(xx_i) = \begin{cases} 2n-1, & i = 1 \\ \dfrac{2n+i-3}{2}, & \text{if } i \text{ is odd}, \quad i \ne 1 \\ \dfrac{3n+i-3}{2}, & \text{if } i \text{ is even} \end{cases}$$

$$f(xy_i) = \begin{cases} 6n+3, & i=1 \\ 7n+6, & i=2 \\ 6n+4, & i=3 \end{cases}$$

$$f(x_i y_i^j) = \begin{cases} \frac{10n+i+4}{2}, & j=1, \quad i \text{ is even} \\ \frac{11n+i+6}{2}, & i \neq n, j=1, i \text{ is odd} \\ \frac{11n+5}{2}, & i=n, j=1 \\ 9n-i+8, & j=2, \quad i \text{ is even} \\ 8n-i+8, & i \neq n, j=2, i \text{ is odd} \\ 8n+7, & i=n, j=2 \\ \frac{14n-i+10}{2}, & j=3, \quad i \text{ is even} \\ \frac{13n-i+8}{2}, & i \neq n, j=3, i \text{ is odd} \\ \frac{13n+9}{2}, & i=n, j=3 \end{cases}$$

$$f(y_i y_{i+1}) = 4n+i$$

$$f(y_i^j y_i^{j+1}) = \begin{cases} 3n+\frac{i}{2}, & j=1, \quad i \text{ is even} \\ \frac{7n+i+2}{2}, & i \neq n, j=1, i \text{ is odd} \\ \frac{7n+1}{2}, & i=n, j=1 \\ \frac{10n-i+6}{2}, & j=2, \quad i \text{ is even} \\ \frac{9n-i+4}{2}, & i \neq n, j=2, i \text{ is odd} \\ \frac{9n+5}{2}, & i=n, j=2 \end{cases}$$

It will be proven that f is the local antimagic vertex total coloring by proving that f is a bijection and neighboring vertices have different weights. First it will be proven that f is a bijection. It is known that $R_f = \{1,2,3,\dots,2n-1\} \cup \{2n, 2n+1, 2n+2, \dots, 3n\} \cup \{3n+1, 3n+2, \dots, 5n+2\} \cup \{5n+3, 5n+4, \dots, 7n+4\} \cup \{7n+5, 7n+6, \dots, 11n+8\}$ so the range of f is $R_f = \{1,2,3,\dots,11n+8\}$. Next, it will be proven that f is an injection. For any $u, v \in V(G)$ and $u \neq v$ applies $f(u) \neq f(v)$, so f is an injective function. Since f is surjection and injection, then f is a bijection.

By function f we get the total vertex weight of graph G. The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2}n(3n+43)+13$$

$$w_t(x_i) = \begin{cases} \frac{49n+25}{2}, & \text{if } i \text{ is odd} \\ \frac{51n+25}{2}, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y_i) = \begin{cases} 19n+11, & \text{if } i \text{ is odd} \\ 26n+17, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y_i^j) = \begin{cases} 19n+11, & \text{if } j \text{ is odd} \\ 26n+17, & \text{if } j \text{ is even} \end{cases}$$

Based on the total vertex weight function it will be shown that neighboring vertices have different weights. Vertex x is adjacent to vertex x_i and vertex y_i so it must have different weight. We assume that $w_t(x) = w_t(x_i) = w_t(y_i)$ or $\frac{1}{2}n(3n+43)+13 = \frac{49n+25}{2} = 19n+11$ if i is odd and $\frac{1}{2}n(3n+43)+13 = \frac{51n+25}{2} = 26n+17$ if i is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex x has a different weight from vertex x_i and y_i.

Next, vertex x_i is adjacent to vertex x_{i+1} and vertex y_i^j. We assume that $w_t(x_i) = w_t(x_{i+1}) = w_t(y_i^j)$ or $\frac{49n+25}{2} = \frac{51n+25}{2} = 19n+11$ if i is odd and $\frac{51n+25}{2} = \frac{49n+25}{2} = 26n+17$ if i is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex x_i has a different weight from vertex x_{i+1} and y_i^j.

Next, vertex y_i is adjacent to vertex y_{i+1}. We assume that $w_t(y_i) = w_t(y_{i+1})$ or $19n+11 = 26n+17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex y_i has a different weight from vertex y_{i+1}.

Last, vertex y_i^j is adjacent to vertex y_i^{j+1}. We assume that $w_t(y_i^j) = w_t(y_i^{j+1})$ or $19n+11 = 26n+17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex y_i^j has a different weight from vertex y_i^{j+1}.

Since each neighboring vertex has different weights and f is a bijection so it can be concluded that f is local antimagic labeling. Then vertices on graph G are colored based on their total vertex weights. This is called the local antimagic vertex total coloring. Because we have 5 values of total weights, then we also have 5 colors.. Therefore, the local antimagic vertex total chromatic number of graph G is $\chi_{lvat}(G) \leq 5$.

Next it will be shown that $\chi_{lvat}(G) \geq 3$. Based on Lemma 1.1. and Lemma 3.1, we have $\chi_{lvat}(G) \geq \chi(G) = 3$. Then $\chi_{lvat}(G) \geq 3$. Because $\chi_{lvat}(G) \leq 5$ and $\chi_{lvat}(G) \geq 3$, it is proven that $3 \leq \chi_{lvat}(G) \leq 5$.

As an illustration, it is presented by Figure 4 which is the local antimagic vertex total coloring of $F_5 \rhd F_3$.

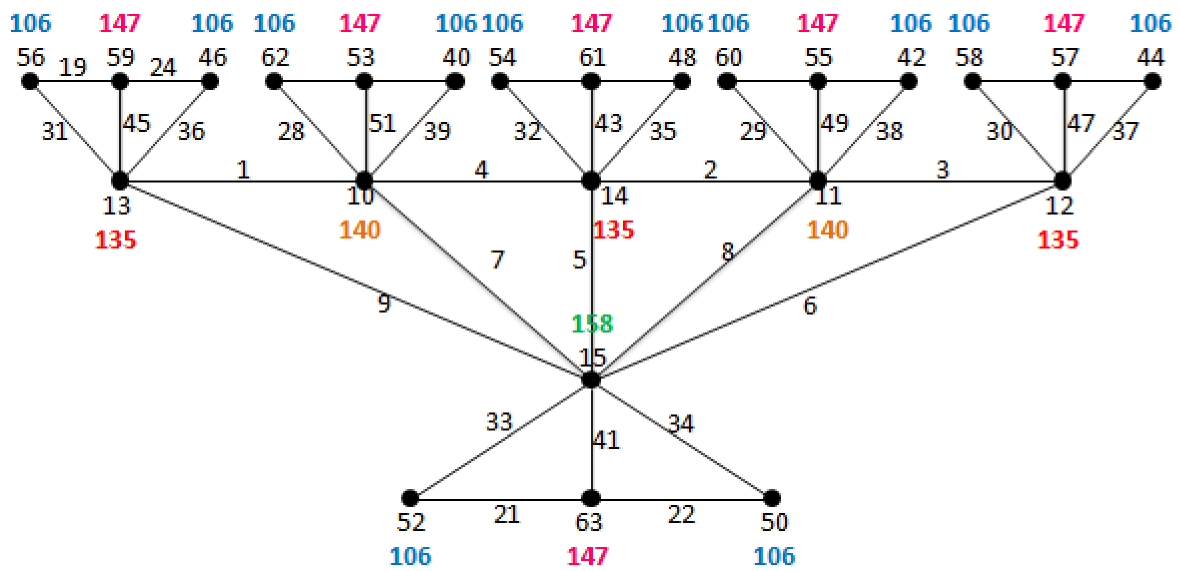

Figure 4. Illustration of the local antimagic vertex total coloring of $F_5 \rhd F_3$.

Case 2. For n even number, it will be proven that $\chi_{lvat}(G) \leq 5$ by labeling graph G using the function of $f: V(G) \cup E(G) \rightarrow \{1,2,3,\ldots,11n+8\}$.

The vertex labeling functions are

$$f(x) = 3n$$

$$f(x_i) = \begin{cases} \frac{4n+i-1}{2}, & \text{if } i \text{ is odd} \\ \frac{5n+i}{2}, & \text{if } i \text{ is even}, \ i \neq n \\ \frac{5n}{2}, & i = n \end{cases}$$

$$f(y_i) = \begin{cases} 9n+7, & i = 1 \\ 11n+8, & i = 2 \\ 9n+11, & i = 3 \end{cases}$$

$$f(y_i^j) = \begin{cases} 11n-i+8, & j = 1, \quad i \text{ is odd} \\ 10n-i+7, & i \neq n, j = 1, i \text{ is even} \\ 10n+7, & i = n, j = 1 \\ 9n+i+7, & j = 2, \quad i \text{ is odd} \\ 10n+i+8, & i \neq n, j = 2, i \text{ is even} \\ 10n+8, & i = n, j = 2 \\ 7n+i+4, & j = 3, \quad i \text{ is odd} \\ 8n+i+5, & i \neq n, j = 3, i \text{ is even} \\ 8n+5, & i = n, j = 3 \end{cases}$$

The edge labeling functions are

$$f(x_i x_{i+1}) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd} \\ \frac{2n-1}{2} & \text{if } i \text{ is even} \end{cases}$$

$$f(xx_i) = \begin{cases} 2n-1, & i = 1 \\ \frac{2n+i-3}{2}, & \text{if } i \text{ is odd}, \quad i \neq 1 \\ \frac{3n+i-4}{2}, & \text{if } i \text{ is even} \end{cases}$$

$$f(xy_i) = \begin{cases} 6n+3, & i = 1 \\ 7n+6, & i = 2 \\ 6n+4, & i = 3 \end{cases}$$

$$f(x_i y_i^j) = \begin{cases} \frac{10n+i+5}{2}, & j = 1, \quad i \text{ is odd} \\ \frac{11n+i+6}{2}, & i \neq n, j = 1, i \text{ is even} \\ \frac{11n+6}{2}, & i = n, j = 1 \\ 9n-i+7, & j = 2, \quad i \text{ is odd} \\ 8n-i+6, & i \neq n, j = 2, i \text{ is even} \\ 8n+6, & i = n, j = 2 \\ \frac{14n-i+9}{2}, & j = 3, \quad i \text{ is odd} \\ \frac{13n-i+8}{2}, & i \neq n, j = 3, i \text{ is even} \\ \frac{13n+8}{2}, & i = n, j = 3 \end{cases}$$

$$f(y_i y_{i+1}) = 4n + i$$

$$f(y_i^j y_i^{j+1}) = \begin{cases} 3n + \frac{i+1}{2}, & j = 1, \quad i \text{ is odd} \\ \frac{7n+i+2}{2}, & i \neq n, j = 1, i \text{ is even} \\ \frac{7n+2}{2}, & i = n, j = 1 \\ \frac{10n-i+5}{2}, & j = 2, \quad i \text{ is odd} \\ \frac{9n-i+4}{2}, & i \neq n, j = 2, i \text{ is even} \\ \frac{9n+4}{2}, & i = n, j = 2 \end{cases}$$

It will be proven that f is the local antimagic vertex total coloring by proving that f is a bijection and neighboring vertices have different weights. First it will be proven that f is a bijection. It is known that $R_f = \{1,2,3,\ldots,2n-1\} \cup \{2n, 2n+1, 2n+2, \ldots, 3n\} \cup \{3n+1, 3n+2, \ldots, 5n+2\} \cup \{5n+3, 5n+4, \ldots, 7n+4\} \cup \{7n+5, 7n+6, \ldots, 11n+8\}$ so the range of f is $R_f = \{1,2,3,\ldots,11n+8\}$. It is clear that range and codomain have the same cardinality so f is a surjective function. Next, it will be proven that f is an injection. For any $u, v \in V(G)$ and $u \neq v$ applies $f(u) \neq f(v)$, so f is an injective function. Since f is surjection and injection, then f is a bijection.

By function f we get total vertex weight of graph G. The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2}n(3n+43) + 13$$

$$w_t(x_i) = \begin{cases} 25n+13, & \text{if } i \text{ is odd} \\ 25n+11, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y_i) = \begin{cases} 19n+11, & \text{if } i \text{ is odd} \\ 26n+17, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y_i^j) = \begin{cases} 19n+11, & \text{if } j \text{ is odd} \\ 26n+17, & \text{if } j \text{ is even} \end{cases}$$

Based on the total vertex weight function it will be shown that neighboring vertices have different weights. Vertex x is adjacent to vertex x_i and vertex y_i so it must have different weight. We assume that $w_t(x) = w_t(x_i) = w_t(y_i)$ or $\frac{1}{2}n(3n+43) + 13 = 25n + 13 = 19n + 11$ if i is odd and $\frac{1}{2}n(3n+43) + 13 = 25n + 11 = 26n + 17$ if i is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex x has a different weight from vertex x_i and y_i.

Next, vertex x_i is adjacent to vertex x_{i+1} and vertex y_i^j. We assume that $w_t(x_i) = w_t(x_{i+1}) = w_t(y_i^j)$ or $25n + 13 = 25n + 11 = 19n + 11$ if i is odd and $25n + 11 = 25n + 13 = 26n + 17$ if i is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex x_i has a different weight from vertex x_{i+1} and y_i^j.

Next, vertex y_i is adjacent to vertex y_{i+1}. We assume that $w_t(y_i) = w_t(y_{i+1})$ or $19n + 11 = 26n + 17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex y_i has a different weight from vertex y_{i+1}.

Last, vertex y_i^j is adjacent to vertex y_i^{j+1}. We assume that $w_t(y_i^j) = w_t(y_i^{j+1})$ or $19n + 11 = 26n + 17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex y_i^j has a different weight from vertex y_i^{j+1}.

Since each neighboring vertex has different weights and f is a bijection so it can be concluded that f is local antimagic labeling. Then vertices on graph G are colored based on their total vertex weights. This is called the local antimagic vertex total coloring. Because we have 5 values of total weights, then we also have 5 colors.. Therefore, the local antimagic vertex total chromatic number of graph G is $\chi_{lvat}(G) \leq 5$.

Next it will be shown that $\chi_{lvat}(G) \geq 3$. Based on Lemma 1.1. and Lemma 3.1, we have $\chi_{lvat}(G) \geq \chi(G) = 3$. Then $\chi_{lvat}(G) \geq 3$. Because $\chi_{lvat}(G) \leq 5$ and $\chi_{lvat}(G) \geq 3$, it is proven that $3 \leq \chi_{lvat}(G) \leq 5$.

As an illustration, it is presented by Figure 5 which is the local antimagic vertex total coloring of $F_4 \rhd F_3$.

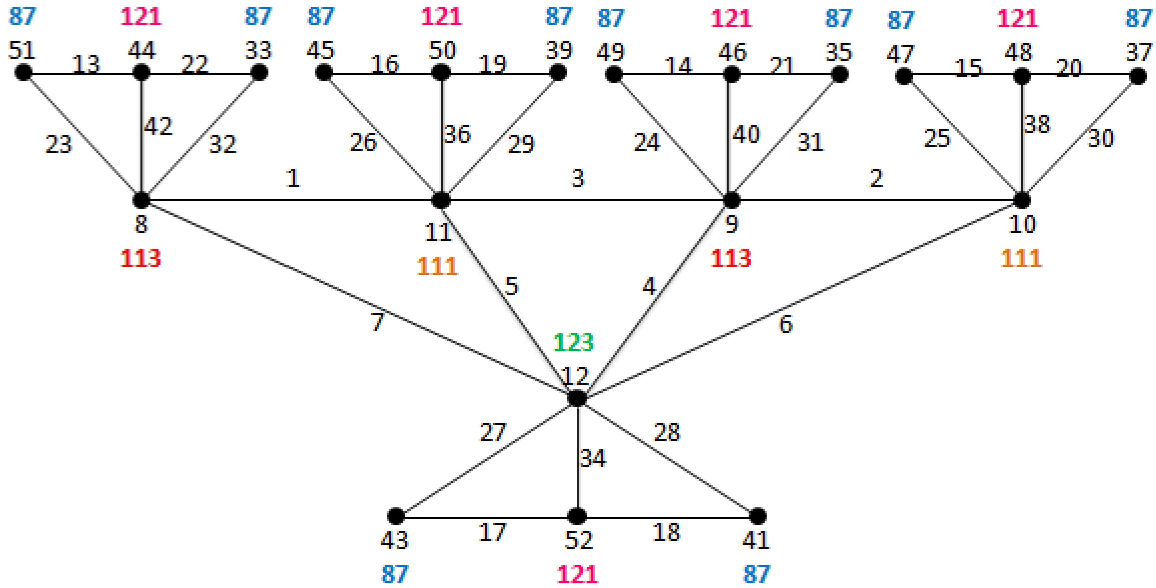

Figure 5. Illustration of the local antimagic vertex total coloring of $F_4 \rhd F_3$.

4. Conclusion

In this study there are two results related to local antimagic vertex total coloring. The resulting theorem is related to the local antimagic vertex total chromatic number of fan graph F_n and fan graph resulting from comb product operation $F_n \rhd F_3$. The local antimagic vertex total chromatic number on F_n, for $n \geq 3$ is $\chi_{lvat}(F_n) = 3$. The local antimagic vertex total chromatic number of $F_n \rhd F_3$ for $n \geq 3$ is $3 \leq \chi_{lvat}(F_n \rhd F_3) \leq 5$.

This result is not yet complete because we don't get the local antimagic vertex total chromatic number of $F_n \rhd F_m$ and graph resulting from comb product operation by grafting the i-th copy of F_m at the center vertex (or non center vertex) to i-th vertex of F_n. It could be an idea to the next research.

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