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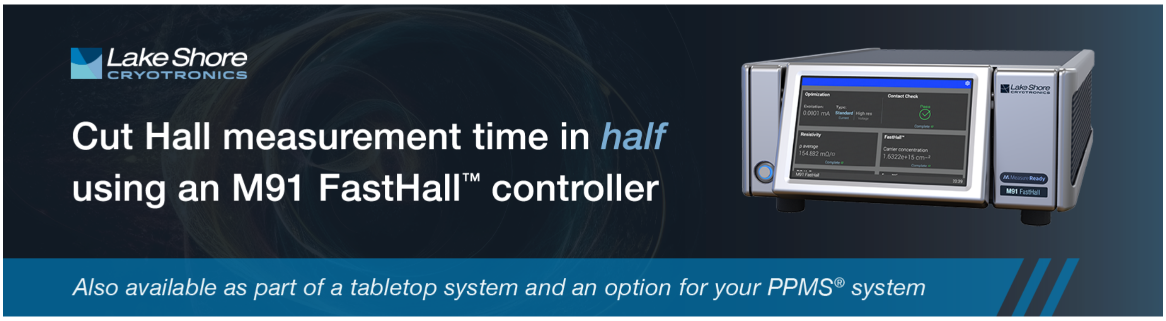

Rainbow Connection Number of k-Corona Product of Graphs

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Abstract. Let G be a nontrivial connected graph. A rainbow path is a path where each edge has different color. A rainbow coloring is a coloring which any two vertices can be joined by at least one rainbow path. A rainbow connection number of a graph, denoted by $rc(G)$, is the smallest number of color required for graph G to be rainbow connected. In this paper, we determine the rainbow connection number of k-corona product of graphs. We get $rc(G\odot^k H) = rc(G) + 3k$ for any integer $k \geq 1$ and G and H are nontrivial connected graphs.

INTRODUCTION

Graph theory is a subject in Mathematics that was first introduced by a Swiss mathematician named Leonhard Euler in 1736. At that time Leonhard Euler tried to find a solution to the problem of the Konigsberg Bridge in the city of Konigsberg (east of Prussia, now Kaliningrad in Russia) which was built at the confluence of two branches of the Pregel river. The first book about graph theory was entitled "Theorie der endlichen und unendlichen Graphen" written by König [1]. One of the subject in graph theory is graph coloring. There are three kinds of graph coloring, namely vertex coloring, edge coloring, and face coloring. One of the concept in edge coloring is a rainbow coloring.

The concept of rainbow connection of a graph was first introduced by Chartrand *et al* [2]. Let G be a nontrivial connected graph. Define a coloring $c: E(G) \rightarrow \{1, 2, \ldots, k\}, k \in N$, where two neighbor edges may have the same color. Let P be a path $u - v$ in G. Path P is called a rainbow path if there are no two edges in P of the same color. A graph G is called rainbow connected if every two different vertices in G are connected by the rainbow path. A rainbow connection number of a graph, denoted by $rc(G)$, is the smallest number of color required for graph G to be rainbow connected. If the rainbow path $u - v$ is geodesic (the length of $u - v$ is the shortest), then G is strongly rainbow-connected and the smallest number of color required for graph G to be strongly rainbow connected is strong rainbow connection number, denoted by $src(G)$. If G is a nontrivial connected graph of size (the number of edges) m whose diameter (the largest distance between two vertices of G) is $diam(G)$, then $diam(G) \leq rc(G) \leq src \leq m$. In this research, Chartrand got the rainbow connection number of some well-known graphs, such as complete graph, tree, cycle, wheel, and complete multipartite graph [2].

The other concept of rainbow connection number is proposed by Krivelevich *et al* [3]. and Li et al [4]. If the initial concept of rainbow connection number is edge-colored graphs, then Krivelevich and Li obtained the concept of rainbow vertex-connected. Vertex-coloured path is a vertex-rainbow if its internal vertices have distinct colors. The rainbow vertex-connection number of G, denoted by $rvc(G)$, is the minimum number of colors in a rainbow vertex-connected vertex coloring of G [3]. Based on the concept of $rc(G)$ and $rvc(G)$, Liu *et al* in 2014 proposed the concept of total rainbow connection in graphs [5].

The recent research on rainbow connection number is the rainbow connection number of $C_m \odot P_n$ and $C_m \odot P_n$ by Maulani *et al* [6], total rainbow connection number on comb product of cycle and path graphs by Hastuti *et al* [7], and

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rainbow connection number and total rainbow connection number of amalgamation results diamond graph (Br_4) and fan graph (F_3) [8].

In this paper, we discuss the rainbow connection number of k-corona product of graphs. The corona operation of two graphs, for example G and H where G is graph of order n, denoted by $G \odot H$, is defined as the graph obtained by taking one copy of G and n copies of graph H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H [9]. For any integer $k \geq 2$, we define the the k-corona products of graphs, denoted by $G \odot^k H$, as $G \odot^k H = (G \odot^{k-1} H) \odot H$ [10]. Figure 1 and Figure 2 are the example of corona product graphs $P_3 \odot C_3$ and $P_3 \odot^2 C_3$, where P_3 is path and C_3 is cycle.

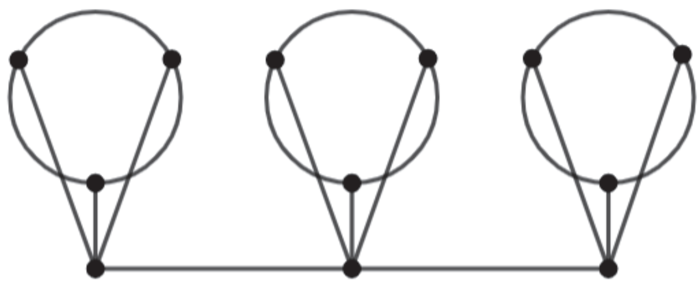

FIGURE 1. Graph $P_3 \odot C_3$

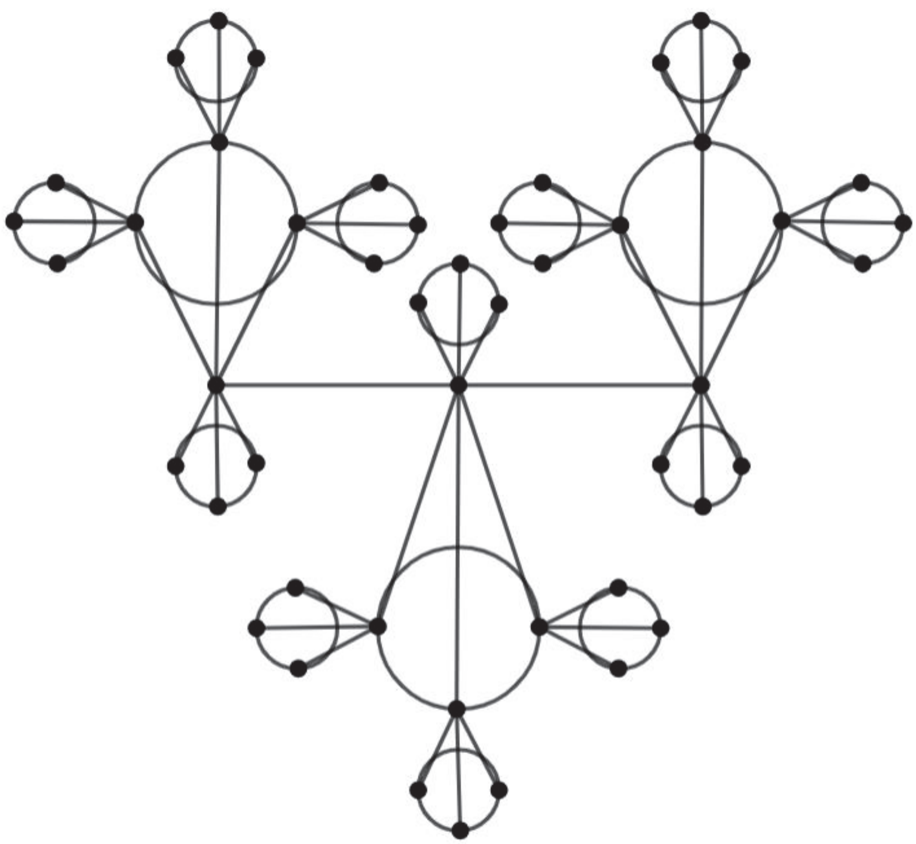

FIGURE 2. Graph $P_3 \odot^2 C_3$

KNOWN RESULT

Definition 1. [2] Let G be a nontrivial connected graph. Define a coloring $c: E(G) \to \{1, 2, \dots, k\}, k \in N$, where two neighbor edges may have the same color. Let P be a path $u - v$ in G. Path P is called a rainbow path if there are no two edges in P of the same color. A graph G is called rainbow connected if every two different vertices in G are connected by the rainbow path. A rainbow connection number of a graph, denoted by $rc(G)$, is the smallest number of color required for graph G to be rainbow connected.

Definition 2. [2] Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a rainbow $u - v$ geodesic in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v (the length of a shortest $u - v$ path in G).

Definition 3. [2] The graph G is strongly rainbow-connected if G contains a rainbow $u - v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a strong rainbow coloring of G. The minimum k for which there exists a coloring $c: E(G) \to \{1, 2, \dots, k\}$ of the edges of G such that G is strongly rainbow-connected is the strong rainbow connection number of G, denoted by $src(G)$.

Conjecture 1. [2] If G is a nontrivial connected graph of size (the number of edges) m and of order (number of vertices) n whose diameter (the largest distance between two vertices of G) is $diam(G)$, then

$$diam(G) \leq rc(G) \leq src(G) \leq m$$
$$diam(G) \leq rc(G) \leq src(G) \leq n - 1$$

Definition 4. [11] A vertex-colored graph G is said to be rainbow vertex-connected if every two vertices of G are connected by a path whose internal vertices have distinct colors, such a path is called a rainbow path. The rainbow vertex-connection number of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected.

Definition 5. [11] A vertex-colored graph G is strongly rainbow vertex-connected, if for every pair u, v of distinct vertices, there exists a rainbow $u - v$ geodesic. The minimum number k for which there exists a k-coloring of G that results in a strongly rainbow vertex-connected graph is called the strong rainbow vertex-connection number of G, denoted by $srvc(G)$.

Conjecture 2. [11] For a nontrivial connected graph G, $diam(G) - 1 \leq rvc(G) \leq srvc(G)$

Definition 6. [5] A total-coloured path is total-rainbow if its edges and internal vertices have distinct colors. The total rainbow k-connection number of G, denoted by $trc(G)$, is the minimum number of colors required to color the edges and vertices of G, so that any two vertices of G are connected by k internally vertex-disjoint total-rainbow paths.

In Table 1, we got the rainbow connection number of several graphs from the previous researchs.

TABLE 1. Rainbow Connection Number of Graphs

Graph	Rainbow Connection Number	
Complete graph (K_n)	$src(K_n) = 1$	[2]
	$trc(K_n) = 1$	[5]
Tree (T) of size m	$rc(T) = m$	[2]
Cycle graph (C_n) for $n \geq 4$	$rc(C_n) = \left\lceil \frac{n}{2} \right\rceil$	[2]
Wheel graph for $n \geq 3$	$rc(W_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7 \end{cases}$	[2]
	$src(W_n) = \left\lceil \frac{n}{3} \right\rceil$	[2]

Graph	Rainbow Connection Number
Complete bipartite graph $(K_{s,t})$ where $1 \leq s \leq t$	$src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$ [2]
Complete bipartite graph $(K_{s,t})$ where $2 \leq s \leq t$	$rc(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil, 4\}$ [2]
Complete k-partite graph $(K_{n_1,n_2,\dots,n_k})$ where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$	$src(K_{n_1,n_2,\dots,n_k}) = \begin{cases} 1 & \text{if } n_k = 1 \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t \\ \lceil \sqrt[s]{t} \rceil & \text{if } s \leq t \end{cases}$ [2]
Corona product graph $C_m \odot P_n$	$rc(C_m \odot P_n) = \begin{cases} 4 & \text{for } m = 3, n \geq 2 \\ \left\lceil \frac{m}{2} \right\rceil + 3 & \text{for } m > 3, n \geq 2 \end{cases}$ [6]
	$src(C_m \odot P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil \cdot 3 + 1 & \text{for } m = 3, n \geq 2 \\ \left\lceil \frac{n}{3} \right\rceil \cdot 3 + \left\lceil \frac{m}{2} \right\rceil & \text{for } m > 3, n \geq 2 \end{cases}$ [6]
Corona product graph $C_m \odot C_n$	$rc(C_m \odot C_n) = \begin{cases} 4 & \text{for } m = 3, n \geq 3 \\ \left\lceil \frac{m}{2} \right\rceil + 3 & \text{for } m > 3, n \geq 3 \end{cases}$ [6]
	$src(C_m \odot P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil \cdot 3 + 1 & \text{for } m = 3, n \geq 3 \\ \left\lceil \frac{n}{3} \right\rceil \cdot 3 + \left\lceil \frac{m}{2} \right\rceil & \text{for } m > 3, n \geq 3 \end{cases}$ [6]
Comb product graph $C_n \rhd C_m$	$trc(C_n \rhd C_m) = n + \left\lceil \frac{n}{2} \right\rceil + 2m - 1$ [7]
Comb product graph $C_n \rhd P_m$	$trc(C_n \rhd P_m) = 2nm - 2n + \left\lceil \frac{n}{2} \right\rceil$ [7]
Comb product graph $P_n \rhd C_m$	$trc(P_n \rhd C_m) = 2n + 2m - 2$ [7]
Comb product graph $P_n \rhd P_m$	$trc(P_n \rhd P_m) = 2nm - n - 1$ [7]

RESEARCH METHOD

The steps used for this research are as follows:

1. Draw the graphs that has to be studied, namely corona product graphs and k-corona product graphs. The graphs that is subjected to corona product operation are path, cycle, and complete graph.
2. Determine the pattern of rainbow coloring of these graphs to get the rainbow connection numbers.
3. Formulate the theorem of rainbow connection number of these graphs and prove it.
4. Set the general formula of rainbow connection number of corona product graphs and k-corona product graphs for any nontrivial connected graphs.

MAIN RESULT

Rainbow Connection Number of $G \odot H$

Before we obtained the rainbow connection number of $G \odot H$, we first get the rainbow connection number of $P_n \odot H$, $C_n \odot H$, and $K_n \odot H$ where P_n is path, C_n is cycle, K_n is complete graph, and H is nontrivial connected graph.

Theorem 1. The rainbow connection number of corona product graphs $P_n \odot H$ where H is a nontrivial connected graph is

$$rc(P_n \odot H) = \begin{cases} 3 & \text{for } n = |V(H)| = 2 \\ n+2 & \text{for another} \end{cases}$$

Proof:

Let H be a graph of order m, $V(P_n \odot H) = \{v_i | i = 1,2, \dots, n\} \cup \{v_i^j | i = 1,2, \dots, n; j = 1,2, \dots, m\}$ and $E(P_n \odot H) = \{v_i v_{i+1} | i = 1,2, \dots, n-1\} \cup \{v_i v_i^j | i = 1,2, \dots, n; j = 1,2, \dots, m\} \cup E(H_1) \cup E(H_2) \cup \dots \cup E(H_n)$ where H_i is the i^{th} copy of H. Since $diam(P_n \odot H) = n + 1$, then $rc(P_n \odot H) \geq n + 1$.

Case 1**.** For $n = m = 2$

If $m = 2$ then $H = P_2$. An edge coloring $c: E(P_2 \odot P_2) \to \{1,2,3\}$ is defined by:

i. $c(v_1 v_2) = 1$
ii. $c(v_1 v_1^1) = c(v_1 v_1^2) = c(v_1^1 v_1^2) = 2$
iii. $c(v_2 v_2^1) = c(v_2 v_2^2) = c(v_2^1 v_2^2) = 3$

By the coloring above there is a rainbow path for each $u, v \in V(P_2 \odot P_2)$ which is shown in Table 2.

TABLE 2. Rainbow Path in $P_2 \odot P_2$

Starting Vertice	End Vertice	Rainbow Path
v_1	v_2	v_1, v_2
v_i where $i = 1,2$	v_i^j where $j = 1,2$	v_i, v_i^j
	v_p^q where $p, q = 1,2$ and $p \neq i$	v_i, v_p, v_p^q
v_i^1 where $i = 1,2$	v_i^2	v_i^1, v_i^2
v_i^j where $i, j = 1,2$	v_p^q where $p, q = 1,2$ and $p \neq i$	v_i^j, v_i, v_p, v_p^q

Next we will prove that 3 is the rainbow connection number of $P_2 \odot P_2$. Note that $rc(P_2 \odot P_2) \geq diam(P_2 \odot P_2) = 3$. So, $rc(P_2 \odot P_2) = 3$.

Case 2**.** For another

An edge coloring $c: E(P_n \odot H) \to \{1,2, \dots, n+2\}$ is defined by:

i. $c(v_i v_{i+1}) = i$ for $i = 1,2, \dots, n-1$
ii. $c(e) = \begin{cases} n & \text{for } e = v_i v_i^j \\ n+1 & \text{for } e = v_i v_i^k \end{cases}$ where v_i^j and v_i^k adjacent; $i = 1,2, \dots, n; j, k = 1,2, \dots, m$
iii. $c(e) = n + 2$ for $e \in E(H)$

By the coloring above there is a rainbow path for each $u, v \in V(P_n \odot H)$ which is shown in Table 3.

TABLE 3. Rainbow Path in $P_n \odot H$

Starting Vertice	End Vertice	Rainbow Path
v_i where $i = 1,2, \dots, n$	v_p where $p = 1,2, \dots, n$ and $p > i$	$v_i, v_{i+1}, \dots, v_p$
	v_i^q where $q = 1,2, \dots, m$	v_i, v_i^q
	v_p^q where $p = 1,2, \dots, n; p > i; q = 1,2, \dots, m$	$v_i, v_{i+1}, \dots, v_p, v_p^q$
v_i^j where $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$	v_i^q where $q = 1,2, \dots, m$; v_i^j and v_i^q are not adjacent	If $c(v_i v_i^j) = c(v_i v_i^q)$ v_i^j, v_i, u, v_i^q where $u \in V(H_i)$, u and v_i^q are adjacent, and $c(v_i u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_i v_i^q)$ v_i^j, v_i, v_i^q

Starting Vertice	End Vertice	Rainbow Path
	v_p^q where $p = 1,2,\dots,n$; $q = 1,2,\dots,m$; $p > i$	If $c(v_i v_i^j) = c(v_p v_p^q)$ $v_i^j, v_i, v_{i+1}, \dots, v_p, u, v_p^q$ where $u \in V(H_p)$, u and v_p^q are adjacent, and $c(v_p u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_i v_i^q)$ $v_i^j, v_i, v_{i+1}, \dots, v_p, v_p^q$

Next we will prove that $n+2$ is the rainbow connection number of $P_n \odot H$. Note that $rc(P_n \odot H) \geq n+1$ and $rc(P_n) = n-1$. Thus there are at least 2 colors in the edges of any $K_1 + H$. If we assume that there are 2 colors in the edges of $K_1 + H$ then there are 2 vertices, namely v_1^i and v_n^j where $c(v_1 v_1^i) = c(v_n v_n^j)$, such that there is no rainbow path from v_1^i to v_n^j.

So, $rc(P_n \odot H) = n+2$ for another.

So, the conclusion is $rc(P_n \odot H) = \begin{cases} 3 & \text{for } n = |V(H)| = 2 \\ n+2 & \text{for another} \end{cases}$

Example 1:
Based on Theorem 1, $rc(P_4 \odot P_4) = 4 + 2 = 6$. The 6-rainbow coloring of graph $P_4 \odot P_4$ is shown in the Figure 3.

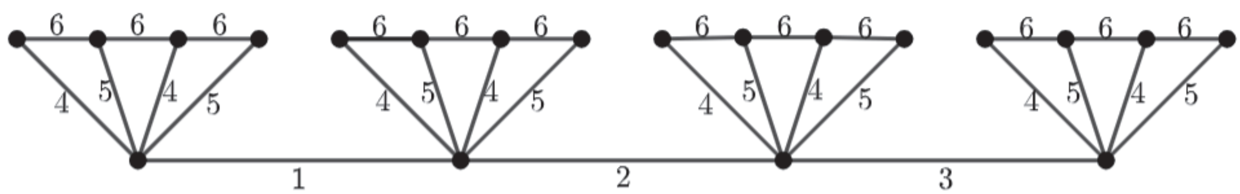

FIGURE 3. Rainbow Coloring of $P_4 \odot P_4$

Theorem 2. The rainbow connection number of corona product graphs $C_n \odot H$ where $n > 3$ and H is a nontrivial connected graph is

$$rc(C_n \odot H) = \left\lceil \frac{n}{2} \right\rceil + 3$$

Proof:

Let H be a graph of order m, $V(C_n \odot H) = \{v_i | i = 1,2,\dots,n\} \cup \{v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $E(C_n \odot H) = \{v_i v_{i+1} | i = 1,2,\dots,n-1\} \cup \{v_1 v_n\} \cup \{v_i v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\} \cup E(H_1) \cup E(H_2) \cup \dots \cup E(H_n)$ where H_i is the i^{th} copy of H. Since $diam(C_n \odot H) = \left\lfloor \frac{n}{2} \right\rfloor + 2$, then $rc(C_n \odot H) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

An edge coloring $c: E(C_n \odot H) \to \left\{1,2,\dots,\left\lceil \frac{n}{2} \right\rceil + 3\right\}$ is defined by:

i. $c(v_i v_{i+1}) = c\left(v_{i+\left\lfloor \frac{n}{2} \right\rfloor} v_{i+\left\lfloor \frac{n}{2} \right\rfloor+1}\right) = i$ for $i = 1,2,\dots,\left\lfloor \frac{n}{2} \right\rfloor$

ii. $c(v_1 v_n) = \left\lceil \frac{n}{2} \right\rceil$

iii. $c(e) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{for } e = v_i v_i^j \\ \left\lceil \frac{n}{2} \right\rceil + 2 & \text{for } e = v_i v_i^k \end{cases}$ where v_i^j and v_i^k adjacent; $i = 1,2,\dots,n; j,k = 1,2,\dots,m$

iv. $c(e) = \left\lceil \frac{n}{2} \right\rceil + 3$ for $e \in E(H)$

By the coloring above there is a rainbow path for each $u, v \in V(C_n \odot H)$ which is shown in Table 4.

TABLE 4. Rainbow Path in $C_n \odot H$

Starting Vertice	End Vertice	Rainbow Path
v_i where $i = 1,2,\dots,n$	v_p where $i < p \le i + \left\lceil \frac{n}{2} \right\rceil$	$v_i, v_{i+1}, \dots, v_p$
	v_p where $i + \left\lceil \frac{n}{2} \right\rceil < p \le n$	$v_p, \dots, v_n, v_1, \dots, v_i$
	v_i^q where $q = 1,2,\dots,m$	v_i, v_i^q
	v_p^q where $i < p \le i + \left\lceil \frac{n}{2} \right\rceil$	$v_i, v_{i+1}, \dots, v_p, v_p^q$
	v_p^q where $i + \left\lceil \frac{n}{2} \right\rceil < p \le n$	$v_p^q, v_p, \dots v_n, v_1, \dots, v_i$
v_i^j where $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$	v_i^q where $q = 1,2,\dots,m$; v_i^j and v_i^q are not adjacent	If $c(v_i v_i^j) = c(v_i v_i^q)$ v_i^j, v_i, u, v_i^q where $u \in V(H_i)$, u and v_i^q are adjacent, and $c(v_i u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_i v_i^q)$ v_i^j, v_i, v_i^q
	v_p^q where $i < p \le i + \left\lceil \frac{n}{2} \right\rceil$	If $c(v_i v_i^j) = c(v_p v_p^q)$ $v_i^j, v_i, v_{i+1}, \dots, v_p, u, v_p^q$ where $u \in V(H_p)$, u and v_p^q are adjacent, and $c(v_p u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_p v_p^q)$ $v_i^j, v_i, v_{i+1}, \dots, v_p, v_p^q$
	v_p^q where $i + \left\lceil \frac{n}{2} \right\rceil < p \le n$	If $c(v_i v_i^j) = c(v_p v_q^q)$ $v_p^q, v_p, \dots, v_n, v_1, \dots, v_i, u, v_i^j$ where $u \in V(H_i)$, u and v_i^j are adjacent, and $c(v_i u) \neq c(v_p v_p^q)$
		If $c(v_i v_i^j) \neq c(v_p v_q^q)$ $v_p^q, v_p, \dots, v_n, v_1, \dots, v_i, v_i^j$

Next we will prove that $\left\lceil \frac{n}{2} \right\rceil + 3$ is the rainbow connection number of $C_n \odot H$. Note that $rc(C_n \odot H) \ge \left\lceil \frac{n}{2} \right\rceil + 2$ and $rc(C_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \ge 4$. Thus there are at least 2 colors in the edges of any $K_1 + H$. If we assume that there are 2 colors in the edges of $K_1 + H$ then there are 2 vertices, namely v_1^i and $v_{\left\lceil \frac{n}{2} \right\rceil}^j$, where $c(v_1 v_1^i) = c\left(v_{\left\lceil \frac{n}{2} \right\rceil} v_{\left\lceil \frac{n}{2} \right\rceil}^j\right)$ such that there is no rainbow path from v_1^i to $v_{\left\lceil \frac{n}{2} \right\rceil}^j$.

So, $rc(C_n \odot H) = \left\lceil \frac{n}{2} \right\rceil + 3$.

Example 2:
Based on Theorem 2, $rc(C_4 \odot P_4) = \left\lceil \frac{4}{2} \right\rceil + 3 = 5$. The 5-rainbow coloring of graph $C_4 \odot P_4$ is shown in the Figure 4.

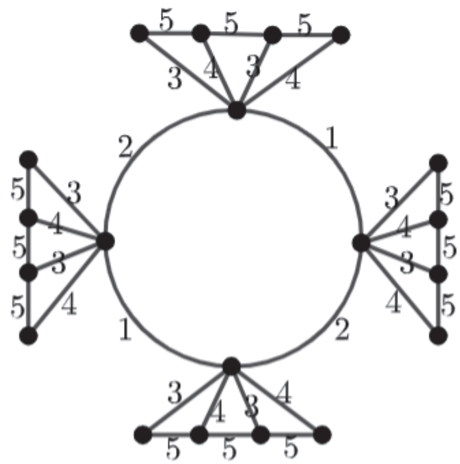

FIGURE 4. Rainbow Coloring of $C_4 \odot P_4$

Theorem 3. The rainbow connection number of corona product graph $K_n \odot H$ where H is a nontrivial connected graph is

$$rc(K_n \odot H) = 4$$

Proof:

Let H be a graph of order m, $V(K_n \odot H) = \{v_i | i = 1,2, \dots, n\} \cup \{v_i^j | i = 1,2, \dots, n; j = 1,2, \dots, m\}$ and $E(K_n \odot H) = \{v_i v_j | i, j = 1,2, \dots, n\} \cup \{v_i v_i^j | i = 1,2, \dots, n; j = 1,2, \dots, m\} \cup E(H_1) \cup E(H_2) \cup \dots \cup E(H_n)$where H_i is the i^{th} copy of H. Since $diam(K_n \odot H) = 3$, then $rc(K_n \odot H) \geq 3$.

An edge coloring $c: E(K_n \odot H) \to \{1,2,3,4\}$ is defined by:

i. $c(v_i v_j) = 1$ for $i, j = 1,2, \dots, n$

ii. $c(e) = \begin{cases} 2 & \text{for } e = v_i v_i^j \\ 3 & \text{for } e = v_i v_i^k \end{cases}$ where v_i^j and v_i^k adjacent; $i = 1,2, \dots, n; j, k = 1,2, \dots, m$

iii. $c(e) = 4$ for $e \in E(H)$

By the coloring above there is a rainbow path for each $u, v \in V(K_n \odot H)$ which is shown in Table 5.

TABLE 5. Rainbow Path in $K_n \odot H$

Starting Vertice	End Vertice	Rainbow Path
v_i where $i = 1,2, \dots, n$	v_p where $p = 1,2, \dots, n$ and $p \neq i$	v_i, v_p
	v_i^q where $q = 1,2, \dots, m$	v_i, v_i^q
	v_p^q where $p = 1,2, \dots, n; p \neq i; q = 1,2, \dots, m$	v_i, v_p, v_p^q
v_i^j where $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$	v_i^q where $q = 1,2, \dots, m$; v_i^j and v_i^q are not adjacent	If $c(v_i v_i^j) = c(v_i v_i^q)$ v_i^j, v_i, u, v_i^q where $u \in V(H_i)$, u and v_i^q are adjacent, and $c(v_i u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_i v_i^q)$ v_i^j, v_i, v_i^q

Starting Vertice	End Vertice	Rainbow Path
	v_p^q where $p = 1,2,\dots,n$; $q = 1,2,\dots,m; p \neq i$	If $c(v_i v_i^j) = c(v_p v_p^q)$ $v_i^j, v_i, v_p, u, v_p^q$ where $u \in V(H_p)$, u and v_p^q are adjacent, and $c(v_p u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_p v_p^q)$ v_i^j, v_i, v_p, v_p^q

Next we will prove that 4 is the rainbow connection number of $K_n \odot H$. Note that $rc(K_n \odot H) \geq 3$ and $rc(K_n) = 1$. Thus there are at least 2 colors in the edges of $K_1 + H$. If we assume that there are 2 colors in the edges of $K_1 + H$ then there are 2 vertices, namely v_i^j and v_k^l, where $i \neq k$ and $c(v_i v_i^j) = c(v_k v_k^l)$, such that there is no rainbow path from v_i^j to v_k^l.

So, $rc(K_n \odot H) = 4$.

Example 3:
Based on Theorem 3, $rc(K_5 \odot P_4) = 4$. The 4-rainbow coloring of graph $K_5 \odot P_4$ is shown in the Figure 5.

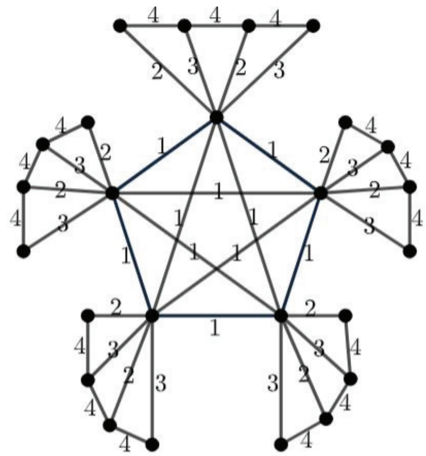

FIGURE 5. Rainbow Coloring of $K_5 \odot P_4$

From the theorems above, we can deduct the rainbow connection number of $G \odot H$ where G and H are nontrivial connected graphs of order more than 2.

Theorem 4. The rainbow connection number of $G \odot H$ where G and H are nontrivial connected graphs of order more than 2 is

$$rc(G \odot H) = rc(G) + 3$$

Proof:

Let G and H be graphs of order n and m respectively, $V(G \odot H) = \{v_i | i = 1,2,\dots,n\} \cup \{v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $E(G \odot H) = E(G) \cup \{v_i v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\} \cup E(H_1) \cup E(H_2) \cup \dots \cup E(H_n)$ where H_i is the i^{th} copy of H. Since $diam(G \odot H) = diam(G) + 2$ then $rc(G \odot H) \geq diam(G) + 2$.

An edge coloring $c: E(G \odot H) \to \{1,2,\dots,rc(G)+3\}$ is defined by:

i. $c: E(G) \to \{1,2,\dots,rc(G)\}$

ii. $c(e) = \begin{cases} rc(G)+1 & \text{if } e = v_i v_i^j \\ rc(G)+2 & \text{if } e = v_i v_i^k \end{cases}$ where v_i^j and v_i^k adjacent; $i = 1,2,\dots,n; j,k = 1,2,\dots,m$

iii. $c: E(H) \to \{rc(G)+3\}$

By coloring above there is a rainbow path for each $u, v \in V(G \odot H)$ which is shown in Table 6.

TABLE 6. Rainbow pPth in $G \odot H$

Starting Vertice	End Vertice	Rainbow Path
v_i where $i = 1,2,\dots,n$	v_p where $p = 1,2,\dots,n$ and $p \neq i$	For any $v_i, v_p \in V(G)$ then there is at least a rainbow path between v_i and v_p
	v_i^q where $q = 1,2,\dots,m$	v_i, v_i^q
	v_p^q where $p = 1,2,\dots,n; p \neq i; q = 1,2,\dots,m$	$v_i - v_p, v_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p
v_i^j where $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$	v_i^q where $q = 1,2,\dots,m$; v_i^j and v_i^q are not adjacent	If $c(v_i v_i^j) = c(v_i, v_i^q)$ v_i^j, v_i, u, v_i^q where $u \in V(H_i)$, u and v_i^q are adjacent, and $c(v_i u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_i v_i^q)$ v_i^j, v_i, v_i^q
	v_p^q where $p = 1,2,\dots,n$; $q = 1,2,\dots,m; p \neq i$	If $c(v_i v_i^j) = c(v_p v_p^q)$ $v_i^j, v_i - v_p, u, v_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p, $u \in V(H_p)$, u and v_p^q are adjacent, and $c(v_p u) \neq c(v_i v_i^j)$
		If $c(v_i v_i^j) \neq c(v_p v_p^q)$ $v_i^j, v_i - v_p, v_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p

Next we will prove that $rc(G) + 3$ is the rainbow connection number of $G \odot H$. Note that $rc(G \odot H) \geq diam(G) + 2$ and $rc(G) \geq diam(G)$. Thus there are at least 2 colors in the edges of $K_1 + H$. If we assume that there are 2 colors in the edges of $K_1 + H$ then there are 2 vertices, namely v_i^j and v_k^l, where $i \neq k, d(v_i, v_k) = diam(G)$, and $c(v_i v_i^j) = c(v_k v_k^l)$, such that there is no rainbow path from v_i^j to v_k^l.

So, $rc(G \odot H) = rc(G) + 3$.

Rainbow Connection Number of $G \odot^k H$

In this section, we get the rainbow connection number of $G \odot^k H$ where G and H are nontrivial connected graphs of order more than 2 and $k \geq 1$.

Theorem 5. The rainbow connection number of k-corona product graphs $G \odot^k H$ where G and H are nontrivial connected graphs of order more than 2, and any integer $k \geq 1$ is

$$rc(G \odot^k H) = rc(G) + 3k$$

Proof:
Use mathematical induction to prove this theorem.

First Step: For $k = 1$ we will prove $rc(G \odot H) = rc(G) + 3$
By Theorem 4, it is proven that $rc(G \odot H) = rc(G) + 3$.

Induction Step: If $rc(G \odot^k H) = rc(G) + 3k$ then we will prove that $rc(G \odot^{k+1} H) = rc(G) + 3(k + 1)$.

Let $V(G\odot^{k+1}H) = V(G\odot^{k}H) \cup \{u_i^j | i = 1,2,\ldots,|V(G\odot^{k}H)|; j = 1,2,\ldots,m\}$.

An edge coloring $c: E(G\odot^{k+1}H) \to \{1,2,\ldots,rc(G)+3k, rc(G)+3k+1, rc(G)+3k+2, rc(G)+3k+3\}$ is defined by:

i. $c: E(G\odot^{k}H) \to \{1,2,\ldots,rc(G)+3k\}$

ii. $c(e) = \begin{cases} rc(G)+3k+1 & \text{if } e = v_i u_i^j \\ rc(G)+3k+2 & \text{if } e = v_i u_i^k \end{cases}$ where u_i^j and u_i^k adjacent, $v_i \in V(G\odot^{k}H)$; $i = 1,2,\ldots,|V(G\odot^{k}H)|; j,k = 1,2,\ldots,m$

iii. $c: \{u_i^j u_k^l | u_i^j, u_k^l \in V(G\odot^{k+1}H) \backslash V(G\odot^{k}H); u_i^j \text{ and } u_k^l \text{ adjacent}\} \to \{rc(G)+3k+3\}$

By coloring above there is a rainbow path for each $u, v \in V(G\odot^{k+1}H)$ which is shown in Table 7.

TABLE 7. Rainbow Path in $G\odot^{k+1}H$

Starting Vertice	End Vertice	Rainbow Path
$v_i \in V(G\odot^{k}H)$	v_p where $p \neq i$	For any $v_i, v_p \in V(G\odot^{k}H)$ then there is at least a rainbow path between v_i and v_p
	$u_i^q \in V(G\odot^{k+1}H) - V(G\odot^{k}H)$	v_i, u_i^q
	$u_p^q \in V(G\odot^{k+1}H) - V(G\odot^{k}H)$; $p \neq i$	$v_i - v_p, u_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p
$u_i^j \in V(G\odot^{k+1}H) - V(G\odot^{k}H)$	$u_i^q \in V(G\odot^{k+1}H) - V(G\odot^{k}H)$ where u_i^j and u_i^q are not adjacent	If $c(v_i u_i^j) = c(v_i, u_i^q)$ u_i^j, v_i, u_i^r, u_i^q where u_i^r and u_i^q are adjacent and $c(v_i u_i^r) \neq c(v_i u_i^j)$
		If $c(v_i u_i^j) \neq c(v_i u_i^q)$ u_i^j, v_i, u_i^q
	$u_p^q \in V(G\odot^{k+1}H) - V(G\odot^{k}H)$ where $p \neq i$	If $c(v_i u_i^j) = c(v_p u_p^q)$ $u_i^j, v_i - v_p, u_p^r, u_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p, u_p^r adjacent to u_p^q and $c(v_p u_p^r) \neq c(v_i u_i^j)$
		If $c(v_i u_i^j) \neq c(v_p u_p^q)$ $u_i^j, v_i - v_p, u_p^q$ where $v_i - v_p$ is a rainbow path between v_i and v_p

Next we will prove that $rc(G)+3k+3$ is the rainbow connection number of $G\odot^{k+1}H$. Note that the edges $E(G\odot^{k+1}H) \backslash E(G\odot^{k}H)$ are given 3 new colors aside from the colors of $E(G\odot^{k}H)$. If we assumed that the new colors are 2 then there are 2 verices, namely u_i^j and u_k^l, where $i \neq k, d(v_i, v_k) = diam(G\odot^{k}H)$, and $c(v_i u_i^j) = c(v_k u_k^l)$, such that there is no rainbow path from u_i^j to u_k^l.

So, $rc(G\odot^{k+1}H) = rc(G) + 3(k+1)$.

From the steps above we can conclude that

$$rc(G\odot^{k}H) = rc(G) + 3k$$

Example 4:
Based on Theorem 5 and Proposition 1, $rc(P_4\odot^2 C_5) = rc(P_4) + 3(2) = 4 - 1 + 6 = 9$. The 9-rainbow coloring of graph $P_4\odot^2 C_5$ is shown in the Figure 6.

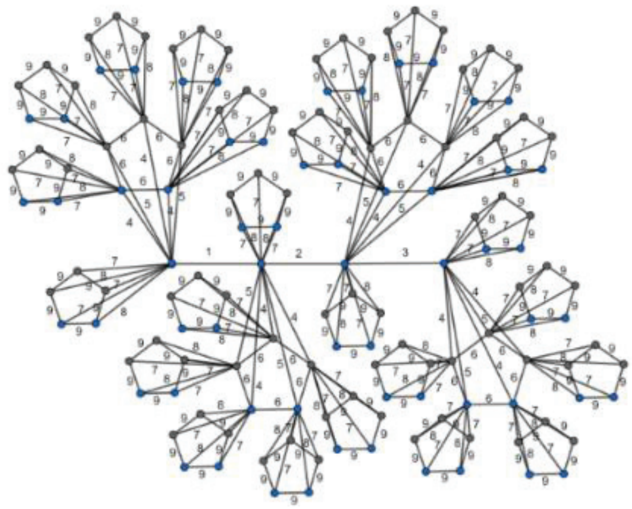

FIGURE 6. Rainbow Coloring of $P_4 \odot^2 C_5$

CONCLUSION

From the discussion that we have in this paper, it can be concluded that

1. The rainbow connection number of corona product graphs are
   a. $rc(P_n \odot H) = \begin{cases} 3 & \text{for } n = |V(H)| = 2 \\ n+2 & \text{for another} \end{cases}$ where P_n is path and H is nontrivial connected graph
   b. $rc(C_n \odot H) = \left\lceil \frac{n}{2} \right\rceil + 3$ where C_n is cycle where $n > 3$ and H is nontrivial connected graph
   c. $rc(K_n \odot H) = 4$ where K_n is complete graph and H is nontrivial connected graph
   d. $rc(G \odot H) = rc(G) + 3$ where G and H are nontrivial connected graphs of order more than 2
2. The rainbow connection number of k-corona product graphs for $k \geq 1$ is
   $$rc(G \odot^k H) = rc(G) + 3k$$
   where G and H are nontrivial connected graphs of order more than 2

In these result, we have developed the previous research into the general formula of rainbow connection number of corona and k-corona product of graphs. In the next research, we suggest other researcher to develop these result for the other concept of rainbow connection number such as strong rainbow connection number or total rainbow connection number.

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