International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137

Volume – IX, Issue – I, January – 2024, PP – 119-125

On commutative characterization of graph operation with respect to complement metric dimension

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Abstract: *Let G be a connected graph with vertex set V(G) and edge set E(G). The distance between vertices u and v in G is denoted by d(u,v), which serves as the shortest path length from u to v. Let W = {w_1,w_2,...,w_k} ⊆ V(G) be an ordered set, and v is a vertex in G. The representation of v with respect to W is an ordered set k-tuple, r(v|W) = (d(v,w_1),d(v,w_2),...,d(w_k)). The set W is called a complement resolving set for G if there are two vertices u,v∈V(G)\W, such that r(u|W)=r(v|W). A complement basis of G is the complement resolving set containing maximum cardinality. The number of vertices in a complement basis of G is called complement metric dimension of G, which is denoted by $\overline{dim}$(G). In general, comb and corona product are non-commutative operations on a graph. However these operations can be commutative in respect to complement metric dimensions. In this study, we get the commutative characterization of corona and comb product, which are $\overline{dim}(G \odot H) = \overline{dim}(H \odot G) \Leftrightarrow 2n + \overline{dim}(K_1 + H) = 2m + \overline{dim}(K_1 + G)$ for $n, m \geq 2$ and $\overline{dim}(G \lhd H). = \overline{dim}(H \lhd G) \Leftrightarrow \overline{dim}(H) - m = \overline{dim}(G) - n$ for $n, m \geq 3$, where n and m are the order of G and H, respectively.*

Keywords: *Comb product operation, complement metric dimension, commutative characterization, corona operation*

I.INTRODUCTION (11 BOLD)

Graph theory are one of the subjects in Discrete Mathematics that have long been known and are widely applied in various fields. Graph theory that has developed rapidly to date started with the problem "Seven Konigsberg Bridges" which Euler successfully solved through his article "Solutio problematicas geometriam situs pertinentis" in 1736 [1].

Until now, graph theory have been used in the study of electronic networks, computer database, discovery of medicine, genetics, architecture, and even the study of computer vision, artificial intelegent and deep learning [5]. One of the topics in graph theory is metric dimension. The metric dimension was first introduced by Harary and Melter at 1976. The study of metric dimension becomes a complate problem, meaning that it is not easy to get the metric dimension of a graph of a certain shape. Therefore, to get the metric dimension of certain shape graphs or certain classes, an analysis of the subclasses is carried out first to make it easier to find the metric dimensions of graphs in general [3]. The metric dimension is the minimum cardinality of resolving sets on graph $\boldsymbol{G}$ which is denoted by $\mathbf{dim}(\boldsymbol{G})$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vertices on a connected graph $\boldsymbol{G}$. The distance from $\boldsymbol{u}$ to $\boldsymbol{v}$ is the length of the shortest path between $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{G}$ denoted by $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v})$ [3].

Other concept of metric dimension is complement metric dimension. Let $\boldsymbol{G}$ be a connected graph that has more than two vertices with the vertex set $\boldsymbol{V}(\boldsymbol{G})$ and the edge set $\boldsymbol{E}(\boldsymbol{G})$. Set $\boldsymbol{S} \subseteq \boldsymbol{V}(\boldsymbol{G})$ is complement resolving set of $\boldsymbol{G}$ if there are two vertices $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{G}) \backslash \boldsymbol{S}$, such that $\boldsymbol{r}(\boldsymbol{u}|\boldsymbol{S}) = \boldsymbol{r}(\boldsymbol{v}|\boldsymbol{S})$. Complement basis of $\boldsymbol{G}$ is complement resolving set that has the maximum cardinality. These cardinality of complement basis is called complement metric dimension, denoted by $\overline{\boldsymbol{dim}}(\boldsymbol{G})$ [9].

Let $*$ be an operation on a graph, the operation $*$ is said to be commutative if $\boldsymbol{G} * \boldsymbol{H} \cong \boldsymbol{H} * \boldsymbol{G}$ for each graph $\boldsymbol{G}$ and $\boldsymbol{H}$. Then, the operation $*$ is said to be commutative with respect to complement metric dimension if $\overline{\mathbf{dim}}(\boldsymbol{G} * \boldsymbol{H}) = \overline{\mathbf{dim}}\,(\boldsymbol{H} * \boldsymbol{G})$, denoted by $\boldsymbol{G} * \boldsymbol{H} \cong_{\overline{\mathbf{dim}}} (\boldsymbol{H} * \boldsymbol{G})$. Research on commutative characterization of corona and comb product operation has been done in [8] related to metric dimension. Meanwhile, research on complement metric dimension of graphs has also been done in [9] by getting complement metric dimension on the path graph $(\boldsymbol{P_n})$, circle graph $(\boldsymbol{C_n})$, star graph $(\boldsymbol{S_n})$, and complete graph $(\boldsymbol{K_n})$. Based on the description above, the purpose of writing this article is to analyze the commutative characterization of corona and comb product operations related to complement metric dimensions.

Theorem 1.1 [8]. $\overline{\mathbf{dim}}(\boldsymbol{P_n}) = \mathbf{1}$, for $\boldsymbol{n} > \mathbf{2}$

Theorem 1.2 [8]. $\overline{\dim}(C_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$ where $n > 2$

Theorem 1.3 [8]. $\overline{\dim}(S_n) = n - 2$, for $n > 2$
Theorem 1.4 [8]. $\overline{\dim}(K_n) = n - 2$, for $n > 2$
Theorem 1.5 [8]. Let G be a connected graph of order n and H be a connected graph of order of order $m \geq 2$. Then

$$\overline{\dim}(G \odot H) = (n-1)(m+1) + \overline{\dim}(K_1 + H)$$

Theorem 1.6 [8]. Let G and H be a connected graph of order of order $n, m \geq 3$. Then

$$\overline{\dim}(G \lhd H) = \begin{cases} \overline{\dim}(H) + (n-1)m & \text{if grafting vertex is a member basis complement of } H \\ m.\overline{\dim}(G) & \text{for another} \end{cases}$$

II.COMPLEMENT METRIC DIMENSION OF CORONA PRODUCT GRAPHS

In this section we get the complement metric dimension of corona product graphs, which are $G \odot P_m$, $G \odot C_m$, $G \odot S_m$, and $G \odot K_m$.

Corollary 2.1 Let G be a connected graph of order n and P_m be a path, then

$$\overline{\dim}(G \odot P_m) = \begin{cases} n(m+1) - 2 & \text{if } m = 2,3 \\ n(m+1) - 3 & \text{if } m > 3 \end{cases}$$

Proof. Based on Theorem 1.5, $\overline{\dim}(G \odot P_m) = (n-1)(m+1) + \overline{\dim}(K_1 + P_m)$. So, first we must get $\overline{\mathbf{dim}}(K_1 + P_m)$. Let $V(K_1 + P_m) = \{v_0, v_1, \dots, v_m\}$ and $E(K_1 + P_m) = \{v_0 v_i | i = 1,2, \dots, m\} \cup \{v_i v_{i+1} | i = 1,2, \dots, m-1\}$.

Case 1. For $m = 2$

Select $W = \{v_0\} \subseteq V(K_1 + P_2)$. Then there are $v_1, v_2 \in V(K_1 + P_2) \backslash W$ such that $r(v_1|W) = r(v_2|W) = (1,1)$. So, W is complement resolving set of.$K_1 + P_2$. Because $|V(K_1 + P_2)| = 3$ and $|W| = 1 = |V(K_1 + P_2)| - 2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1 + P_2) = 1$.

Case 2. For $m = 3$

Select $W = \{v_1, v_3\} \subseteq V(K_1 + P_3)$. Then there are $v_0, v_2 \in V(K_1 + P_3) \backslash W$ such that $r(v_0|W) = r(v_2|W) = (1,1)$. So, W is complement resolving set of $K_1 + P_3$. Because $|V(K_1 + P_3)| = 4$ and $|W| = 2 = |V(K_1 + P_3)| - 2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1 + P_3) = 2$.

Case 3. For $m > 3$

Select $W = \{v_0, v_4, v_5, v_6, \dots, v_m\} \subseteq V(K_1 + P_m)$. Then there are $v_1, v_2 \in V(K_1 + P_m) \backslash W$ such that $r(v_1|W) = r(v_2|W) = (1,2,2, \dots, 2)$. So, W is complement resolving set of $K_1 + P_m$.

Next, it will be proved that W with $|W| = m - 2$ is the complement resolving set with maximum cardinality. Take any set $W' \subseteq V(K_1 + P_m)$ so that $|W'| > m - 2$. Suppose $|W'| = m - 1$. Then there are two vertices outside of W'. As for the possibilities of W' are:

i. If $W' = V(K_1 + P_m) \backslash \{v_0, v_i\}$ where $i = 1,2, \dots, m$ then $r(v_0|W') \neq r(v_i|W')$ because $d(v_0, v_{i+2}) = 1$ while $d(v_i, v_{i+2}) = 2$
ii. If $W' = V(K_1 + P_m) \backslash \{v_i, v_{i+1}\}$ then $r(v_i|W') \neq r(v_{i+1}|W')$ because $d(v_i, v_{i+2}) = 2$ while $d(v_{i+1}, v_{i+2}) = 1$
iii. If $W' = V(K_1 + P_m) \backslash \{v_1, v_i\}$ with $i = 3,4, \dots, m-1$ then $r(v_1|W') \neq r(v_i|W')$ because $d(v_1, v_{i+1}) = 2$ while $d(v_i, v_{i+1}) = 1$
iv. If $W' = V(K_1 + P_m) \backslash \{v_k, v_l\}$ with $k, l = 2,3, \dots, m-1, k < l$ then $r(v_k|W') \neq r(v_l|W')$ because $d\,(v_k, v_{k-1}) = 1$ while $d(v_l, v_{k-1}) = 2$

Based on the possibilities above, it can be concluded that W' with $|W'| = m - 1$ is not the complement resolving set of $K_1 + P_m$. Therefore W is the complement resolving set with maximum cardinality. So $\overline{dim}(K_1 + P_m) = m - 2$

From **Case 1**, **Case 2**, and **Case 3**, we obtained $\overline{dim}(K_1 + P_m) = \begin{cases} m-1 & \text{if } m = 2,3 \\ m-2 & \text{if } m > 3 \end{cases}$ Therefore

$$\begin{aligned} \overline{dim}(G \odot P_m) &= (n-1)(m+1) + \begin{cases} m-1 & \text{if } m = 2,3 \\ m-2 & \text{if } m > 3 \end{cases} \\ &= nm + n - m - 1 + \begin{cases} m-1 & \text{if } m = 2,3 \\ m-2 & \text{if } m > 3 \end{cases} \\ &= \begin{cases} nm + n - 2 & \text{if } m = 2,3 \\ nm + n - 3 & \text{if } m > 3 \end{cases} \end{aligned}$$

Thus $\overline{dim}(G \odot P_m) = \begin{cases} n(m+1) - 2 & \text{if } m = 2,3 \\ n(m+1) - 3 & \text{if } m > 3 \end{cases}$ ∎

Corollary 2.2 Let G is a connected graph of order n and C_m is a cycle graph, then

$$\overline{dim}(G \odot C_m) = \begin{cases} n(m+1)-2 & \text{if } m = 3,4 \\ n(m+1)-4 & \text{if } m > 4 \end{cases}$$

Proof. Based on Theorem 1.5, $\overline{dim}(G \odot C_m) = (n-1)(m+1) + \overline{dim}(K_1 + C_m)$. So, first we determine $\overline{dim}(K_1 + C_m)$. Suppose $V(K_1 + C_m) = \{v_0, v_1, \dots, v_m\}$ and $E(K_1 + C_m) = \{v_0 v_i | i = 1,2,\dots,m\} \cup \{v_i v_{i+1} | i = 1,2,3,\dots,m-1\} \cup \{v_1 v_m\}$

Case 1. For $m = 3$

Select $W = \{v_0, v_1\} \subseteq V(K_1 + C_3)$. Then there are $v_2, v_3 \in V(K_1 + C_3) \backslash W$ such that $r(v_2|W) = r(v_3|W) = (1,1)$. So, W is complement resolving set of $K_1 + C_3$. Because $|V(K_1 + C_3)| = 4$ and $|W| = 2 = |V(K_1 + C_3)| - 2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1 + C_3) = 2$.

Case 2. For $m = 4$

Select $W = \{v_0, v_1, v_3\} \subseteq V(K_1 + C_4)$. Then there are $v_2, v_4 \in V(K_1 + C_4) \backslash W$ such that $r(v_2|W) = r(v_4|W) = (1,1,1)$. So, W is complement resolving set of $K_1 + C_4$. Because $|V(K_1 + C_4)| = 5$ and $|W| = 3 = |V(K_1 + C_4)| - 2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1 + C_4) = 3$.

Case 3. For $m > 4$

Select $W = \{v_0, v_5, v_6, \dots, v_m\} \subseteq V(K_1 + C_m)$. Then there are $v_2, v_3 \in (K_1 + C_m) \backslash W$ such that $r(v_2|W) = r(v_3|W) = (1,2,2,\dots,2)$. So, W is complement resolving set of $K_1 + C_m$.

Next, it will be proved that W with $|W| = m-3$ is complement resolving set with maximum cardinality. Take any set $W' \subseteq V(K_1 + C_m)$ so that $|W'| > m-3$. Suppose $|W'| = m-2$. Then there are three vertices outside of W'. As for the possibilities of W' are:

i. If $W' = V(K_1 + C_m) \backslash \{v_0, v_i, v_{i+1}\}$ where $i = 1,2,\dots,m-1$ then
   - $r(v_0|W') \neq r(v_i|W')$ because $d(v_0, v_{i+2}) = 1$ while $d(v_i, v_{i+2}) = 2$
   - $r(v_0|W') \neq r(v_{i+1}|W')$ because $d(v_0, v_{i+3}) = 1$ while $d(v_{i+1}, v_{i+3}) = 2$
   - $r(v_i|W') \neq r(v_{i+1}|W')$ because $d(v_i, v_{i+2}) = 2$ while $d(v_{i+1}, v_{i+2}) = 1$

ii. If $W' = V(K_1 + C_m) \backslash \{v_0, v_i, v_j\}$ where $i, j = 1,2,\dots,m$ and $j > i+1$ then
   - $r(v_0|W') \neq r(v_i|W')$ because $d(v_0, v_{i+2}) = 1$ while $d(v_i, v_{i+2}) = 2$
   - $r(v_0|W') \neq r(v_j|W')$ because $d(v_0, v_{j+2}) = 1$ while $d(v_j, v_{j+2}) = 2$
   - $r(v_i|W') \neq r(v_j|W')$ because $d(v_i, v_{j+1}) = 2$ while $d(v_j, v_{j+1}) = 1$

iii. If $W' = V(K_1 + C_m) \backslash \{v_i, v_{i+1}, v_{i+2}\}$ where $i = 1,2,\dots,m-2$ then
   - $r(v_i|W') \neq r(v_{i+1}|W')$ because $d(v_i, v_{i-1}) = 1$ while $d(v_{i+1}, v_{i-1}) = 2$
   - $r(v_i|W') \neq r(v_{i+2}|W')$ because $d(v_i, v_{i+3}) = 2$ while $d(v_{i+2}, v_{i+3}) = 1$
   - $r(v_{i+1}|W') \neq r(v_{i+2}|W')$ because $d(v_{i+1}, v_{i+3}) = 2$ while $d(v_{i+2}, v_{i+3}) = 1$

iv. If $W' = V(K_1 + C_m) \backslash \{v_i, v_{i+1}, v_j\}$ where $i, j = 1,2,\dots,m-1$ and $j > i+2$ then
   - $r(v_i|W') \neq r(v_{i+1}|W')$ because $d(v_i, v_{i-1}) = 1$ while $d(v_{i+1}, v_{i-1}) = 2$
   - $r(v_i|W') \neq r(v_j|W')$ because $d(v_i, v_{j+1}) = 2$ while $d(v_j, v_{j+1}) = 1$
   - $r(v_{i+1}|W') \neq r(v_j|W')$ because $d(v_{i+1}, v_{j+1}) = 2$ while $d(v_j, v_{j+1}) = 1$

v. If $W' = V(K_1 + C_m) \backslash \{v_i, v_j, v_k\}$ where $i, j, k = 1,2,\dots,m$ and $j > i+1$, $k > j+1$ then
   - $r(v_i|W') \neq r(v_j|W')$ because $d(v_i, v_{j+1}) = 2$ while $d(v_j, v_{j+1}) = 1$
   - $r(v_i|W') \neq r(v_k|W')$ because $d(v_i, v_{k+1}) = 2$ while $d(v_k, v_{k+1}) = 1$
   - $r(v_j|W') \neq r(v_k|W')$ because $d(v_j, v_{k+1}) = 2$ while $d(v_k, v_{k+1}) = 1$

Based on the possibilities above, it can be concluded that W' with $|W'| = m-2$ is not the complement resolving set of $K_1 + C_m$. Therefore W is the complement resolving set with maximum cardinality. So $\overline{dim}(K_1 + C_m) = m-3$.

From **Case 1**, **Case 2**, and **Case 3**, we obtained $\overline{dim}(K_1 + C_m) = \begin{cases} m-1 & \text{if } m = 3,4 \\ m-3 & \text{if } m > 4 \end{cases}$ Therefore

$$\begin{aligned} \overline{dim}(G \odot C_m) &= (n-1)(m+1) + \begin{cases} m-1 & \text{if } m = 3,4 \\ m-3 & \text{if } m > 4 \end{cases} \\ &= nm + n - m - 1 + \begin{cases} m-1 & \text{if } m = 3,4 \\ m-3 & \text{if } m > 4 \end{cases} \\ &= \begin{cases} nm + n - 2 & \text{if } m = 3,4 \\ nm + n - 4 & \text{if } m > 4 \end{cases} \end{aligned}$$

Thus $\overline{dim}(G \odot C_m) = \begin{cases} n(m+1)-2 & \text{if } m = 3,4 \\ n(m+1)-4 & \text{if } m > 4 \end{cases}$ ■

Corollary 2.3 Let G be a connected graph of order n and K_m is a complete graph, then $\overline{dim}(G\odot K_m) = n(m+1)-2$

Proof. Based on Theorem 1.5, $\overline{dim}(G\odot K_m) = (n-1)(m+1) + \overline{dim}(K_1+K_m)$. Therefore, first we determine $\overline{dim}(K_1+K_m)$. Suppose $V(K_1+K_m) = \{v_0, v_1, \dots, v_m\}$ and $E(K_1+K_m) = \{v_iv_j | i,j = 0,1,2,\dots,m\}$. Select $W = \{v_2, v_3, \dots, v_m\} \subseteq V(K_1+K_m)$. Then there are $v_0, v_1 \in V(K_1+K_m)\backslash W$ such that $r(v_0|W) = r(v_1|W) = (1,1,\dots,1)$. So, W is complement resolving set of K_1+K_m. Because $|V(K_1+K_m)| = m+1$ and $|W| = m-1 = |V(K_1+K_m)|-2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1+K_m) = m-1$. Threfore

$$\begin{aligned}\overline{dim}(G\odot K_m) &= (n-1)(m+1)+m-1\\ &= nm+n-m-1+m-1\\ &= nm+n-2\end{aligned}$$

Thus $\overline{dim}(G\odot K_m) = n(m+1)-2$. ■

Corollary 2.4 Let G be a connected graph of order n and S_m is a star graph, then $\overline{dim}(G\odot S_m) = n(m+2)-2$

Proof. Based on Theorem 1.5 $\overline{dim}(G\odot S_m) = (n-1)(m+2) + \overline{dim}(K_1+S_m)$. So, first we determine $\overline{dim}(K_1+S_m)$. Suppose $V(K_1+S_m) = \{c, v_0, v_1, \dots, v_m\}$ and $E(K_1+S_m) = \{cv_i | i = 0,1,2,\dots,m\} \cup \{v_0v_i | i = 1,2,3,\dots,m\}$. Select $W = \{c, v_0, v_3, v_4, \dots, v_m\} \subseteq V(K_1+S_m)$. Then there are $v_1, v_2 \in V(K_1+S_m)\backslash W$ such that $r(v_1|W) = r(v_2|W) = (1,1,2,2,\dots,2)$. So, W is complement resolving set of K_1+S_m. Because $|V(K_1+S_m)| = m+2$ and $|W| = m = |V(K_1+S_m)|-2$ is the maximum cardinality of complement resolving set, then $\overline{dim}(K_1+S_m) = m$. Therefore

$$\overline{dim}(G\odot S_m) = (n-1)(m+2)+m = nm+2n-m-2+m = nm+2n-2$$

Thus $\overline{dim}(G\odot S_m) = n(m+2)-2$. ■

III. COMPLEMENT METRIC DIMENSION OF COMB PRODUCT GRAPH

In this section we get the complement metric dimension of comb product graphs, which are $G \lhd P_m$, $G \lhd C_m$, $G \lhd S_m$, and $G \lhd K_m$.

Corollary 3.1 Let G is a connected graph of order $n \geq 3$ and P_m is a path graph with $m \geq 3$, then

$$\overline{dim}(G \lhd P_m) = \begin{cases}(n-1)m+1 & \text{if the grafting vertex of } P_m \text{ is not end vertex}\\ m\,\overline{dim}\,(G) & \text{if the grafting vertex of } P_m \text{ is the end vertex}\end{cases}$$

Proof. Let $V(G \lhd P_m) = \{v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $E(G \lhd P_m) = E(G) \cup \{v_i^j v_i^{j+1} | i = 1,2,\dots,n; j = 1,2,\dots,m-1\}$.

Case 1. If the grafting vertex of P_m is not end vertex

For example, the grafting vertex of P_m is v_i^k where $k = 2,3,\dots,m-1$. Select $W = \{v_1^k\} \cup \{v_i^j | i = 2,3,\dots,n; j = 1,2,\dots,m\} \subseteq V(G \lhd P_m)$. Then there are $v_1^{k-1}, v_1^{k+1} \in V(G \rhd P_m)\backslash W$ such that $r(v_1^{k-1}|W) = r(v_1^{k+1}|W)$ for $i = 2,3,\dots,n; j = 1,2,\dots,m$. So, W is complement resolving set of $G \rhd P_m$.

Furthermore, it will be shown that W with $|W| = (n-1)m+1$ is the complement resolving set with maximum cardinality. Take any set $W' \subseteq V(G \lhd P_m)$ so that $|W'| > (n-1)m+1$. Suppose $W' = W \cup \{a\}$ where $a \in \{v_i^j | j \neq k\}$. Then the possibilities of W':

i. If $W' = W \cup \{v_1^l\}$ where $l = 1, m$, then for $p, q \neq k, l$ apply $r(v_1^p|W') \neq r(v_1^q|W')$ because $d(v_1^p, v_1^l) \neq d(v_1^q, v_1^l)$.

ii. If $W' = W \cup \{v_1^l\}$ where $l \neq 1, k, m$, then for $p, q \neq k$ apply $r(v_1^p|W') \neq r(v_1^q|W')$ because $(d(v_1^p, v_1^k), d(v_1^p, v_1^l) \neq (d(v_1^q, v_1^k), d(v_1^q, v_1^l)$.

Based on the possibilities above, it can be concluded that W with $|W'| > (n-1)m+1$ is not the complement resolving set of $G \lhd P_m$. Therefore W is the complement resolving set with maximum cardinality. So , $\overline{dim}(G \lhd P_m) = (n-1)m+1$ if the grafting vertex of P_m is not end vertex.

Case 2. If the grafting vertex of P_m is the end vertex

Let the grafting vertex of P_m is v_i^1. If W_G is basis complement of graph G and $W_G = \{w_i | i = 1,2,\dots,\overline{dim}\,(G)\}$ then $W_G \subseteq V(G) = \{v_i^1 | i = 2,3,\dots,n\}$ and there is $v_p^1, v_q^1 \in V(G)\backslash W_G$ such that $r(v_p^1|W_G) = r(v_q^1|W_G)$. Select $W = W_G \cup \{v_i^j | v_i^1 = w_i; j = 2,3,\dots,m\} \subseteq V(G \rhd P_m)$, then $v_p^1, v_q^1 \in V(G \rhd P_m)\backslash W$ and $r(v_p^1|W) = r(v_q^1|W)$ because $d(v_p^1, v_i^j) = d(v_q^1, v_i^j)$ for $v_i^1 = w_1$ and $j = 2,3,\dots,m$. So, W is complement resolving set in graph $G \lhd P_m$.

Furthermore, it will be proved that W where $|W| = m.\overline{dim}(G)$ is the complement resolving set with maximum cardinality. Take any set $W' \subseteq V(G \lhd P_m)$ so that $|W'| > m.\overline{dim}(G)$. Suppose $W' = W \cup \{a\}$ with $a \in V(G \lhd P_m)\backslash W$. Then the possibilities of W':

i. If $W' = W \cup \{v_l^1\}$ where $v_l^1 \notin W_G$ then W' is not basis complement of graph $G \lhd P_m$ because W_G is complement basis of G so it is not possible to have a complement resolving set in graph G which has a cardinality greater so W_G.

ii. If $W' = W \cup \{v_l^j\}$ with $v_l^1 \notin W_G$ and $j = 2,3,\dots,m$, then W' is not basis complement of $G \lhd P_m$ because v_l^j is located in one branch with v_l^1 which is also not a member of the basis of graph G.

Based on these possibilities, it can be concluded that W' with $|W'| > m.\overline{dim}(G)$ is not complement resolving set of graph $G \lhd P_m$. Therefore W is the complement resolving set with maximum cardinality. So , $\overline{dim}(G \lhd P_m) = m.\overline{dim}(G)$ if the grafting vertex of P_m is the end vertex.

Based on **Case 1** and **Case 2**, we obtained

$$\overline{dim}(G \lhd P_m) = \begin{cases} (n-1)m + 1 & \text{if the grafting vertex of } P_m \text{ is not end vertex} \\ m\,\overline{dim}\,(G) & \text{if the grafting vertex of } P_m \text{ is the end vertex} \end{cases} \blacksquare$$

Corollary 3.2 Let G be a connected graph of order $n \geq 3$ and C_m is a cycle graph with $m \geq 3$, then

$$\overline{dim}(G \lhd C_m) = \begin{cases} (n-1)m + 1 & \text{if } m \text{ is odd} \\ (n-1)m + 2 & \text{if } m \text{ is even} \end{cases}$$

Proof. Let $V(G \lhd C_m) = \{v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $E(G \lhd P_m) = E(G) \cup \{v_i^j v_i^{j+1} | i = 1,2,\dots,n; j = 1,2,\dots,m-1\} \cup \{v_i^1 v_i^m | i = 1,2,\dots,n\}$.

Case 1. If m is odd

Suppose the grafting vertex on C_m is v_i^1 where $i = 1,2,\dots,n$. Select $W = \{v_1^1\} \cup \{v_i^j | i = 2,3,\dots,n; j = 1,2,\dots,m\} \subseteq V(G \lhd C_m)$. Then there are $v_1^m, v_1^2 \in V(G \lhd C_m)\backslash W$ such that $r(v_1^m|W) = r(v_1^2|W)$. So, W is complement resolving set of $G \lhd C_m$.

Furthermore, it will be proved that W with $|W| = (n-1)m + 1$ is the complement resolving set with maximum cardinality. Take any set $W' \subseteq V(G \lhd C_m)$ so that $|W'| > (n-1)m + 1$. Suppose $W' = W \cup \{a\}$ with $a \in \{v_1^j | j = 2,3,\dots,m\}$. Then W' is not a complement resolving set of $G \lhd C_m$ because $r(v_1^p|W') \neq r(v_1^q|W')$ for $p, q \neq 1, j$.

Based on the evidence, it can be concluded that W with $|W'| > (n-1)m + 1$ is not the complement resolving set of $G \lhd C_m$. Therefore W is the complement resolving set with maximum cardinality. So, $\overline{dim}(G \lhd C_m) = (n-1)m + 1$ if m is odd.

Case 2. If m is even

Suppose the grafting vertex on C_m is v_i^1 where $i = 1,2,\dots,n$. Select $W = \left\{v_1^1, v_1^{\frac{1}{2}m+1}\right\} \cup \{v_i^j | i = 2,3,\dots,n; j = 1,2,\dots,m\} \subseteq V(G \lhd C_m)$. Then there are $v_1^m, v_1^2 \in V(G \lhd C_m)\backslash W$ such that $r(v_1^m|W) = r(v_1^2|W)$. So, W is the complement resolving set of $G \lhd C_m$.

Furthermore, it will be proved that W with $|W| = (n-1)m + 2$ is the complement resolving set with maximum cardinality. Take any set $W' \subseteq V(G \rhd C_m)$ so that $|W'| > (n-1)m + 2$. Suppose $W' = W \cup \{a\}$ with $a \in \{v_1^j | j = 2,3,\dots,m\}$. Then W' is not a complement resolving set of $G \lhd C_m$ because $r(v_1^p|W') \neq r(v_1^q|W')$ for $p, q \neq 1, j, \frac{1}{2}m + 1$

Based on the evidence, it can be concluded that W with $|W'| > (n-1)m + 2$ is not the complement resolving set of $G \lhd C_m$. Therefore W is the complement resolving set with maximum cardinality. So, $\overline{dim}(G \lhd C_m) = (n-1)m + 2$ if m is even.

Based on **Case 1** and **Case 2**, we obtained $\overline{dim}(G \lhd C_m) = \begin{cases} (n-1)m + 1 & \text{if } m \text{ is odd} \\ (n-1)m + 2 & \text{if } m \text{ is even} \end{cases}$ ■

Corollary 3.3 Let G be a connected graph of order n and K_m is a complete graph with $m \geq 3$, then $\overline{dim}(G \lhd K_m) = mn - 2$

Proof. Let $V(G \lhd K_m) = \{v_j^i | i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $E(G \lhd K_m) = E(G) \cup \{v_i^j v_i^k | i = 1,2,\dots,n; j,k = 1,2,\dots,m\}$. Suppose the grafting vertex on K_m is v_i^1 with $i = 1,2,\dots,n$. Select $W = \{v_1^j | j = 1,2,\dots,m-2\} \cup \{v_i^j | i = 2,3,\dots,n; j = 1,2,\dots,m\} \subseteq V(G \rhd K_m)$. Then there are $v_1^m, v_1^{m-1} \in V(G \lhd K_m)\backslash W$ such that $r(v_1^m|W) = r(v_1^{m-1}|W)$. So, W is the complement resolving set of $G \lhd K_m$. Because $|V(G \lhd K_m)| =$

mn and $|W| = mn - 2 = |V(G \lhd K_m)| - 2$ is the complement resolving set with maximum cardinality, then $\overline{dim}(G \lhd K_m) = mn - 2$. ■

Corollary 3.4 Let G be a connected graph of order $n \geq 3$ and S_m is a star graph with $m \geq 2$, then $\overline{dim}(G \lhd S_m) = n(m+1) - 2$.

Proof. Let $V(G \lhd S_m) = \{v_i^j | i = 1,2,\dots,n; j = 0,1,2,\dots,m\}$ and $E(G \lhd S_m) = E(G) \cup \{v_i^0 v_i^j | i = 1,2,\dots,n; j = 1,2,\dots,m\}$. Suppose the grafting vertex on S_m is is v_i^0 with $i = 1,2,\dots,n$. Select $W = \{v_1^j | j = 0,1,2,\dots,m-2\} \cup \{v_i^j | i = 2,3,\dots,n; j = 0,1,2,\dots,m\} \subseteq V(G \lhd S_m)$. Then there are $v_1^m, v_1^{m-1} \in V(G \lhd S_m) \backslash W$ such that $r(v_1^m|W) = r(v_1^{m-1}|W)$. So, W is the complement resolving set of $G \rhd S_m$. Because $|V(G \lhd S_m)| = (m+1)n$ and $|W| = n(m+1) - 2 = |V(G \lhd S_m)| - 2$ is the complement resolving set with maximum cardinality, then $\overline{dim}(G \lhd S_m) = n(m+1) - 2$. ■

IV.COMMUTATIVE CHARACTERIZATION OF CORONA AND COMB PRODUCTS OF GRAPHS WITH RESPECT TO COMPLEMENT METRIC DIMENSION

In this section we get the characterization of corona and comb products of graphs with respect to complement metric dimension.

Corollary 4.1 Let G and H be connected graphs with order n and m where $n, m \geq 2$. Then

$$\overline{dim}(G \odot H) = \overline{dim}(H \odot G) \Leftrightarrow 2n + \overline{dim}(K_1 + H) = 2m + \overline{dim}(K_1 + G)$$

Proof.

($\Rightarrow$) Suppose $\overline{dim}(G \odot H) = \overline{dim}(H \odot G)$. Based onTheorem 1.5,

$$\overline{dim}(G \odot H) = \overline{dim}(H \odot G)$$
$$(n-1)(m+1) + \overline{dim}(K_1 + H) = (m-1)(n+1) + \overline{dim}(K_1 + G)$$
$$nm + n - m - 1 + \overline{dim}(K_1 + H) = nm - n + m - 1 + \overline{dim}(K_1 + G)$$
$$n - m + \overline{dim}(K_1 + H) = m - n + \overline{dim}(K_1 + G)$$
$$2n + \overline{dim}(K_1 + H) = 2m + \overline{dim}(K_1 + G)$$

($\Leftarrow$) Suppose $2n + \overline{dim}(K_1 + H) = 2m + \overline{dim}(K_1 + G)$. Then $\overline{dim}(K_1 + H) = 2m - 2n + \overline{dim}(K_1 + G)$. Based on Theorem 1.5, $\overline{dim}(G \odot H) = (n-1)(m+1) + \overline{dim}(K_1 + H)$. Thus

$$\begin{aligned}\overline{dim}(G \odot H) &= (n-1)(m+1) + 2m - 2n + \overline{dim}(K_1 + G)\\ &= nm + n - m - 1 + 2m - 2n + \overline{dim}(K_1 + G)\\ &= nm - n + m - 1 + \overline{dim}(K_1 + G) \dots \text{(i)}\end{aligned}$$

Based on Theorem 1.5 too,

$\overline{dim}(H \odot G) = (m-1)(n+1) + \overline{dim}(K_1 + G) = nm - n + m - 1 + \overline{dim}(K_1 + G)$ ... (ii)

It can be seen that Equation (i) is equal to Equation (ii) therefore $\overline{dim}(G \odot H) = \overline{dim}(H \odot G)$. ■

Corollary 4.2 Let G and H be connected graphs other than path with order n and m where $n, m \geq 3$. Then

$$\overline{dim}(G \lhd H). = \overline{dim}(H \lhd G) \Leftrightarrow \overline{dim}(H) - m = \overline{dim}(G) - n$$

Proof.

($\Rightarrow$) Suppose $\overline{dim}(G \lhd H) = \overline{dim}(H \lhd G)$. Based on Theorem 1.6,

$$\overline{dim}(G \lhd H) = \overline{dim}(H \lhd G)$$
$$\overline{dim}(H) + (n-1)m = \overline{dim}(G) + (m-1)n$$
$$\overline{dim}(H) + nm - m = \overline{dim}(G) + nm - n$$
$$\overline{dim}(H) - m = \overline{dim}(G) - n$$

($\Leftarrow$) Suppose $\overline{dim}(H) - m = \overline{dim}(G) - n$. Then $\overline{dim}(H) = \overline{dim}(G) + m - n$. Based on Theorem 1.6, $\overline{dim}(G \lhd H) = \overline{dim}\, H + (n-1)m$. Thus

$$\begin{aligned}\overline{dim}(G \lhd H) &= \overline{dim}(G) + m - n + (n-1)m\\ &= \overline{dim}(G) + m - n + nm - m\\ &= \overline{dim}(G) - n + nm \dots \text{(i)}\end{aligned}$$

Based onTheorem 1.6 too,

$$\begin{aligned}\overline{dim}(H \lhd G) &= \overline{dim}(G) + (m-1)n\\ &= \overline{dim}(G) + nm - n \dots \text{(ii)}\end{aligned}$$

It can be seen that Equation (i) is equal to Equation (ii) so $\overline{dim}(G \lhd H) = \overline{dim}(H \lhd G)$. ■

V.CONCLUSION

In this paper we obtained two results related to the commutative characterization of corona and comb products of graphs with respect to complement metric dimension which are:

i. Commutative characterization of corona products of graphs with respect to complement metric dimension is $\overline{dim}(G \odot H) = \overline{dim}(H \odot G) \Leftrightarrow 2n + \overline{dim}(K_1 + H) = 2m + \overline{dim}(K_1 + G)$ where n and m are the order of G and H and $nm \geq 2$.

ii. Commutative characterization of comb products of graphs with respect to complement metric dimension is $\overline{dim}(G \lhd H). = \overline{dim}(H \lhd G) \Leftrightarrow \overline{dim}(H) - m = \overline{dim}(G) - n$ where n and m are the order of G and H and $n, m \geq 3$.

Acknowledgements

Thank you to Department Mathematics, Mathematics and Science Faculty, and LPPM Universitas Islam Madura for the assistance in this research.

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