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Abstract. Let G be a connected graph. A vertex w is said to strongly resolve a pair u, v of vertices of G if there exists some shortest $u - w$ path containing v or some shortest $v - w$ path containing u. A set W of vertices is a local strong resolving set for G if every pair of adjacent vertices of G is strongly resolved by some vertex of W. The smallest cardinality of a local strong resolving set for G is called the local strong metric dimension of G. In this paper we studied the local strong metric dimension in the join of graphs. We use the path, cycle, complete, and star graphs in this studies.

1. Introduction

Graph theory is a subject in mathematics which first introduced by a Swiss mathematician named Leonard Euler in 1736, as an effort to solve the Konigsberg Bridge problem [4]. Graphs are used to represent discrete objects and the relationships between those objects. The visual representation of a graph is to represent objects as vertices (nodes) and the relationships between objects as edges [2]. One of the studies that continue to develop in graph theory is matric dimension.

The concept of metric dimension in graph theory has been introduced separately by Slater in 1975 and by Harrary and Melter in 1976. Slater relates the problem of metric dimensions to determine the number of sonar detection tools in a network [8] while Harrary defines metric dimensions through the set of differentiators. This concept can be used to distinguish each point on a graph G by determining its representation againt the set of vertices from G [1]. The concept of metric dimension then was developed into local metric dimensions by Okamoto et al. [7]. This concept states that every adjacent vertices on a graph has different representation in regard to local differentiators [3]. Okamoto et al. obtained the result that a nontrivial connected graph G has local metric dimension $n - 1$ if and only if G is complete graphs. Also, graph G has a local metric dimension of 1 if and only if G is bipartite graphs.

Another developed concept of metric dimension is strong metric dimension. This concept is found by Sebo and Tannier in 2004. A vertex w is said to strongly resolve a pair u, v of vertices of G if there exists some shortest $u - w$ path containing v or some shortest $v - w$ path containing u [6]. A set that contains those vertices is called strong resolving set of G. The strong metric dimension of G is the smallest cardinality of strong resolving set, denoted by $sdim(G)$. The concept of strong metric dimension then developed into local strong metric dimension [10]. A set W of vertices is a local strong resolving set for G if every pair of adjacent vertices of G is strongly resolved by some vertex of W. The smallest cardinality of a local strong resolving set for G is called the local strong metric dimension of G, denoted by $\dim_{ls}(G)$. Several results regarding local strong metric dimension has been found in path graph, star graph, complete graph, cycle graph, and graph resulting from corona product.

Research on local strong metric dimension has been carried out by Susilowati et al. [9] on $\boldsymbol{k}$ −level corona product graphs. The next research was conducted by Sutardji et al. [10] on cartesian product graph. Furthermore, research on the graph resulting from join operation has been carried out by Kuziak et al. [5] to get the strong metric dimension.

Therefore, in this paper we discuss the local strong metric dimension in the join of graphs.

Theorem 1.1 [10] *Let* $\boldsymbol{G}$ *be a connected graph of order* $\boldsymbol{n \geq 2}$, *then*

i. $\boldsymbol{dim_{ls}(G) = 1}$ *if and only if* $\boldsymbol{G}$ *is bipartite graph*

ii. $\boldsymbol{dim_{ls}(G) = n - 1}$ *if and only if* $\boldsymbol{G}$ *is complete graph*

Theorem 1.2 [9] *Let* $\boldsymbol{P_n}$ *be a path graph,* $\boldsymbol{C_n}$ *be a cycle, and* $\boldsymbol{S_n}$ *be a star graph, then*

1. $\boldsymbol{dim_{ls}(P_n) = 1}$
2. $\boldsymbol{dim_{ls}(C_n) = \begin{cases} 1 & if\ n\ is\ even \\ 2 & if\ n\ is\ odd \end{cases}}$
3. $\boldsymbol{dim_{ls}(S_n) = 1}$

2. Local strong metric dimension of $K_1 + G$

In this section, we obtained the local strong metric dimension in the join between a trivial graph (K_1) and G, where G is one of the following graphs, which are path (P_n), cycle (C_n), complete graph (K_n), and star graph (S_n).

Before we discuss the local strong metric dimension in those graphs, we need to know the symbol of $I[u, v]$ which denotes the shortest path between vertex u and vertex v. If $W \subseteq V(G)$ is a local strong resolving set, then a vertex $w \in W$ is said to strongly resolve an adjacent pair u, v of vertices of G if $u \in I[v, w]$ or $v \in I[u, w]$. We do not have to check every adjacent vertices in G, just adjacent vertices in G other than W.

Generally speaking, if G is graph resulting from join operation then $diam(G) = 2$, where $diam(G)$ is diameter of G which is $diam(G) = \max_{u,v \in V(G)} d(u, v)$. So, in the join of graphs, $d(u, v) = 2$ if u and v are not adjacent vertices.

Theorem 2.1 *Let* G *be a join graph* $K_1 + P_n$, *where* P_n *is a path, then* $dim_{ls}(G) = 2$ *if* $2 \leq n \leq 5$ *and*

$$dim_{ls}(G) = \begin{cases} \left\lceil \frac{n-3}{4} \right\rceil + 1 & for\ (n-3) mod\ 4 < 3 \\ \left\lceil \frac{n-3}{4} \right\rceil + 1 & for\ (n-3) mod\ 4 = 3 \end{cases}$$

if $n > 5$, *where* $[x] = m$ *for* $m < x < m + 1$ *and* m *is integer.*

Proof.

Let $V(G) = \{c\} \cup \{v_i | i = 1,2, \ldots, n\}$ and $E(G) = \{cv_i | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | i = 1,2, \ldots, n - 1\}$.

Case 1. For $2 \leq n \leq 5$

Let $W = \{c, v_{\frac{n+1}{2}}\}$ if n is odd and $W = \{c, v_{\frac{n}{2}}\}$ if n is even. Then for every pair of adjacent vertices in $V(G) \backslash W$:

i. $(v_i, v_{i+1}) \rightarrow v_{i+1} \in I[v_i, v_{i+2}]$ or $v_i \in I[v_{i-1}, v_{i+1}]$

ii. $(c, v_i) \rightarrow c \in I[v_i, v_j]$ where v_i and v_j are not adjacent

So, W is a local strong resolving set of G.

Next, we will prove that W is the local strong resolving set with the smallest cardinality. Take any set $W' \subseteq V(G)$ where $|W'| < 2$, so there are two option of W' which are:

i. If $W' = \{c\}$ then for a pair adjacent vertice $(v_i, v_{i+1}) \rightarrow v_{i+1} \notin I[v_i, c]$ and $v_i \notin I[v_{i+1}, c]$. So $W' = \{c\}$ is not a local strong resolving set of G.

ii. If $W' = \{v_i\}$ then for a pair adjacent vertice $(c, v_{i+1}) \rightarrow v_{i+1} \notin I[c, v_i]$ and $c \notin I[v_{i+1}, v_i]$. So $W' = \{v_i\}$ is not a local strong resolving set of G.

Based on the argument above, W' where $|W'| = 1$ is not a local strong resolving set of G. Therefore, W is the local strong resolving set with the smallest cardinality. Thus, $\dim_{ls}(G) = 2$ if $2 \leq n \leq 5$.

Case 2. For $n > 5$

Let $W = \left\{v_{4i+3} \middle| i = 0,1,\dots,\left\lceil\frac{n-3}{4}\right\rceil\right\}$ if $(n-3) mod\ 4 < 3$ and $W = \left\{v_{4i+3} \middle| i = 0,1,\dots,\left\lceil\frac{n-3}{4}\right\rceil\right\} \cup \{v_n\}$ if $(n-3)mod\ 4 < 3$. Then for every pair of adjacent vertices in $V(G)\backslash W$:

i. $(v_1, v_2) \rightarrow v_2 \in I[v_1, v_3]$
ii. $(v_{4i+4}, v_{4i+5}) \rightarrow v_{4i+4} \in I[v_{4i+5}, v_{4i+3}]$ for $i = 0,1,\dots,\left\lceil\frac{n-3}{4}\right\rceil$
iii. $(v_{4i+5}, v_{4i+6}) \rightarrow v_{4i+6} \in I[v_{4i+5}, v_{4(i+1)+3}]$ for $i = 0,1,\dots,\left\lceil\frac{n-3}{4}\right\rceil$

So, W is a local strong resolving set of G.

Next, we will prove that W is the local strong resolving set with the smallest cardinality. Take any set $W' \subseteq V(G)$ where $|W'| < |W|$. Suppose $W' = W - \{a\}$ where $a \in W$, then:

i. If $W' = W - \{v_3\}$ then for a pair adjacent vertice $(v_1, v_2) \rightarrow v_2 \notin I[v_1, w]$ and $v_1 \notin I[v_2, w]$, where $w \in W'$. So W' is not a local strong resolving set of G.
ii. If $W' = W - \{v_{4k+3}\}$ where $k = 1,2,\dots,\left\lceil\frac{n-3}{4}\right\rceil$ then for a pair adjacent vertice $(v_{4k+1}, v_{4k+2}) \rightarrow v_{4k+1} \notin I[v_{4k+2}, w]$ and $v_{4k+2} \notin I[v_{4k+1}, w]$, where $w \in W'$. So W' is not a local strong resolving set of G.

Based on the argument above, W' where $|W'| < |W|$ is not a local strong resolving set of G. Therefore, W is the local strong resolving set with the smallest cardinality. Thus, $\dim_{ls}(G) = \begin{cases} \left\lceil\frac{n-3}{4}\right\rceil + 1 & \text{for } (n-3) mod\ 4 < 3 \\ \left\lceil\frac{n-3}{4}\right\rceil + 1 & \text{for } (n-3) mod\ 4 = 3 \end{cases}$ if $n > 5$. ∎

***Theorem 2.2** Let G be a join graph $K_1 + C_n$, where C_n is a cycle graph, then $dim_{ls}(G) = 3$ if $3 \le n \le 5$ and $dim_{ls}(G) = \left\lceil\frac{n-1}{4}\right\rceil + 1$ if $n > 5$, where $\lceil x \rceil = m$ for $m < x < m + 1$ and m is integer.*

Proof.

Let $V(G) = \{c\} \cup \{v_i | i = 1,2,\dots,n\}$ and $E(G) = \{cv_i | i = 1,2,\dots,n\} \cup \{v_i v_{i+1} | i = 1,2,\dots,n-1\} \cup \{v_1 v_n\}$.

Case 1. For $3 \le n \le 5$

Let $W = \{c, v_1, v_{\frac{n+1}{2}}\}$ if n is odd and $W = \{c, v_1, v_{\frac{n}{2}}\}$ if n is even. Then for every pair of adjacent vertices in $V(G)\backslash W$:

i. $(v_1, v_n) \rightarrow v_1 \in I[v_n, v_2]$ or $v_n \in I[v_1, v_{n-1}]$
ii. $(v_i, v_{i+1}) \rightarrow v_{i+1} \in I[v_i, v_{i+2}]$ or $v_i \in I[v_{i-1}, v_{i+1}]$
iii. $(c, v_i) \rightarrow c \in I[v_i, v_j]$ where v_i and v_j are not adjacent

So, W is a local strong resolving set of G.

Next, we will prove that W is the local strong resolving set with the smallest cardinality. Take any set $W' \subseteq V(G)$ where $|W'| < 3$, so there are three option of W' which are:

i. If $W' = \{c, v_i\}$ then for a pair adjacent vertice $(v_{i+1}, v_{i+2}) \rightarrow v_{i+1} \notin I[v_{i+2}, c]$ and $v_{i+2} \notin I[v_{i+1}, c]$. So $W' = \{c, v_i\}$ is not a local strong resolving set of G.
ii. If $W' = \{v_i, v_{i+1}\}$ then for a pair adjacent vertice $(c, v_{i+2}) \rightarrow v_{i+2} \notin I[c, v_i]$ and $c \notin I[v_{i+2}, v_{i+1}]$. So $W' = \{v_i, v_{i+1}\}$ is not a local strong resolving set of G.
iii. If $W' = \{v_i, v_j\}$, where v_i and v_j are not adjacent, then for a pair adjacent vertice $(c, v_{j+1}) \rightarrow v_{j+1} \notin I[c, v_i]$ and $c \notin I[v_{j+1}, v_j]$. So $W' = \{v_i, v_j\}$ is not a local strong resolving set of G.

Based on the argument above, W' where $|W'| = 2$ is not a local strong resolving set of G. Therefore, W is the local strong resolving set with the smallest cardinality. Thus, $\dim_{ls}(G) = 3$ if $3 \le n \le 5$.

Case 2. For $n > 5$

Let $W = \left\{v_{4i+1} \middle| i = 0,1,\dots,\left\lceil\frac{n-1}{4}\right\rceil\right\}$. Then for every pair of adjacent vertices in $V(G)\backslash W$:

i. $(v_{4i+2}, v_{4i+3}) \rightarrow v_{4i+2} \in I[v_{4i+3}, v_{4i+1}]$ for $i = 0,1,\dots,\left\lceil\frac{n-1}{4}\right\rceil$
ii. $(v_{4i+3}, v_{4i+4}) \rightarrow v_{4i+4} \in I[v_{4i+3}, v_{4(i+1)+1}]$ for $i = 0,1,\dots,\left\lceil\frac{n-1}{4}\right\rceil$

iii. $(c, v_i) \rightarrow c \in I[v_i, v_{4i+1}]$ for $i = 1,2, \ldots, \left\lceil\frac{n-1}{4}\right\rceil$

So, W is a local strong resolving set of G.

Next, we will prove that W is the local strong resolving set with the smallest cardinality. Take any set $W' \subseteq V(G)$ where $|W'| < |W|$. Suppose $W' = W - \{a\}$ where $a \in W$, then if $W' = W - \{v_{4k+1}\}$ where $k = 0,1, \ldots, \left\lceil\frac{n-1}{4}\right\rceil$ then for a pair adjacent vertice $(v_{4k+2}, v_{4k+3}) \rightarrow v_{4k+2} \notin I[v_{4k+3}, w]$ and $v_{4k+3} \notin I[v_{4k+2}, w]$, where $w \in W'$. So W' is not a local strong resolving set of G.

Based on the argument above, W' where $|W'| < |W|$ is not a local strong resolving set of G. Therefore, W is the local strong resolving set with the smallest cardinality. Thus, $\dim_{ls}(G) = \left\lceil\frac{n-1}{4}\right\rceil + 1$ if $n > 5$. ■

***Theorem 2.3** Let G be a join graph $K_1 + K_n$, where K_n is a complete graph with order $n \geq 2$, then $dim_{ls}(G) = n$*

Proof.

Let $V(G) = \{c\} \cup \{v_i | 1 \leq i \leq n\}$ and $E(G) = \{cv_i | i = 1,2, \ldots, n\} \cup \{v_i v_j | i, j = 1,2, \ldots, n\}$.

Graph $G = K_1 + K_n$ is isomorphic with graph K_{n+1}. Based on Theorem 1.1, $\dim_{ls}(K_{n+1}) = n + 1 - 1 = n$. Thus $\dim_{ls}(G) = n$. ■

***Theorem 2.4** Let G be a join graph $K_1 + S_n$, where S_n is a star graph with order $n \geq 2$, then $dim_{ls}(G) = 2$*

Proof.

Let $V(G) = \{c\} \cup \{v_i | i = 0,1, \ldots, n\}$ and $E(G) = \{cv_i | i = 0,1, \ldots, n\} \cup \{v_0 v_i | i = 1,2, \ldots, n\}$.

Suppose $W = \{c, v_0\}$, then for every pair of adjacent vertices in $V(G)$:

i. $(c, v_i) \rightarrow c \in I[v_i, c]$ for $i = 0,1, \ldots, n$

ii. $(v_0, v_i) \rightarrow v_0 \in I[v_i, v_0]$ for $i = 1,2, \ldots, n$

So, W is a local strong resolving set of G.

Next, we will prove that W is the local strong resolving set with the smallest cardinality. Take any set $W' \subseteq V(G)$ where $|W'| < 2$, so there are three option of W' which are:

i. If $W' = \{c\}$ then for a pair adjacent vertice $(v_0, v_i) \rightarrow v_i \notin I[v_0, c]$ and $v_0 \notin I[v_i, c]$. So $W' = \{c\}$ is not a local strong resolving set of G.

ii. If $W' = \{v_0\}$ then for a pair adjacent vertice $(c, v_i) \rightarrow v_i \notin I[c, v_0]$ and $c \notin I[v_i, v_0]$. So $W' = \{v_0\}$ is not a local strong resolving set of G.

iii. If $W' = \{v_k\}$ where $k = 1,2, \ldots, n$, then for a pair adjacent vertice $(c, v_0) \rightarrow v_0 \notin I[c, v_k]$ and $c \notin I[v_0, v_k]$. So $W' = \{v_k\}$ is not a local strong resolving set of G.

Based on the argument above, W' where $|W'| = 1$ is not a local strong resolving set of G. Therefore, W is the local strong resolving set with the smallest cardinality. Thus, $\dim_{ls}(G) = 2$. ■

3. Local strong metric dimension of $G + H$

In this section, we get the local strong metric dimension in the join of graphs, $G + H$, with G and H be a connected graph. First, we discuss the properties of the local strong basis of $G + H$. Local strong basis is the local strong resolving set with the minimum cardinality and this cardinality is called the local strong metric dimension.

***Lemma 3.1** Let $W = W_1 \cup W_2$ be the local strong basis of $G + H$ where $W_1 \subseteq V(G)$ and $W_2 \subseteq V(H)$, where $|V(G)| \geq 2$ and $|V(H)| \geq 2$, then $\dim_{ls}(G + H) \geq 3$, with $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$.*

Proof. Let $V(G) = \{u_i | i = 1,2, \ldots, n\}$ and $V(H) = \{v_i | i = 1,2, \ldots, m\}$. First, we assume that $W_1 = \emptyset$ or $W_2 = \emptyset$. If $W_1 = \emptyset$ then $W = W_2$. So, for a pair of adjacent vertices $(u_i, u_j) \rightarrow u_i \notin I[u_j, v_k]$ and $u_j \notin I[u_i, v_k]$ where $v_k \in W_2$. On the other hand, if $W_2 = \emptyset$ then $W = W_1$. So, for a pair of adjacent vertices $(v_i, v_j) \rightarrow v_i \notin I[v_j, u_k]$ and $v_j \notin I[v_i, u_k]$ where $u_k \in W_1$. Therefore, $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$.

Next, if we let $W = \{u_k, v_k\}$ then for a pair of adjacent vertices (u_i, v_j) where u_i adjacent with u_k and v_i adjacent with v_k applies $v_j \notin I[u_i, u_k]$ and $u_i \notin I[v_j, v_k]$. So, W with two elements is not the local strong basis of $G + H$. Thus $\dim_{ls}(G + H) \geq 3$. ■

Lemma 3.2 *Let the join of two connected graphs, $G + H$, with $|V(G)| = n$ and $|V(H)| = m$ and $V(G + H) = \{u_i|i = 1, 2, \ldots, n\} \cup \{v_i|i = 1, 2, \ldots, m\}$ then $d(u_i, u_j) = d(v_i, v_j) = 2$ if u_i is not adjacent to u_j and v_i is not adjacent to v_j. Furthermore, $d(u_i, v_j) = 1$.*

Proof. Since $V(G + H) = \{u_i|i = 1, 2, \ldots, n\} \cup \{v_i|i = 1, 2, \ldots, m\}$ then $u_i v_j \in E(G + H)$. Thus $d(u_i, v_j) = 1$. Next, since u_i is not adjacent to u_j and v_i is not adjacent to v_j then $d(u_i, u_j) \neq 1$ and $d(v_i, v_j) \neq 1$. Because $diam(G + H) = 2$ then $d(u_i, u_j) = d(v_i, v_j) = 2$. ■

Using the lemma above, we get the local strong metric dimension of $G + H$ which is stated in the following theorem.

Theorem 3.1. *Let G be a connected graph with order n ≥ 2 and H be a connected graph with order m ≥ 2 then*

$$\dim_{ls}(G+H) = \begin{cases} \dim_{ls}(K_1+G) + \dim_{ls}(K_1+H) & \text{if } diam(G) > 2 \text{ and } diam(H) > 2 \\ \dim_{ls}(K_1+G) + \dim_{ls}(H) & \text{if } diam(G) > 2 \text{ and } diam(H) \leq 2 \\ \dim_{ls}(G) + \dim_{ls}(K_1+H) & \text{if } diam(G) \leq 2 \text{ and } diam(H) > 2 \\ \dim_{ls}(G) + \dim_{ls}(H) + 1 & \text{if } diam(G) \leq 2 \text{ and } diam(H) \leq 2 \\ & \text{or } \dim_{ls}(G) = \dim_{ls}(H) = 1 \end{cases}$$

Proof. Let $V(G) = \{u_i|i = 1,2, \ldots, n\}, V(H) = \{v_i|i = 1,2, \ldots, m\}, V(G + H) = V(G) \cup V(H)$, and $E(G + H) = E(G) \cup E(H) \cup \{u_i v_j|i = 1,2, \ldots, n; j = 1,2, \ldots, m\}$. Suppose $W = W_1 \cup W_2$ where $W_1 \subseteq V(G)$ and $W_2 \subseteq V(H)$. In order to get the minimum cardinality of W, we get four cases which are:

Case 1. If $diam(G) > 2$ and $diam(H) > 2$

According to Lemma 3.2, $d(u_i, u_j) = d(v_i, v_j) = 2$ if u_i is not adjacent to u_j and v_i is not adjacent to v_j. In order to satisfy this lemma in this case, $|W_1| = \dim_{ls}(K_1 + G)$ and $|W_2| = \dim_{ls}(K_1 + H)$. So, $\dim_{ls}(G + H) = \dim_{ls}(K_1 + G) + \dim_{ls}(K_1 + H)$ if $diam(G) > 2$ and $diam(H) > 2$.

Case 2. If $diam(G) > 2$ and $diam(H) \leq 2$

According to Lemma 3.2, $d(u_i, u_j) = d(v_i, v_j) = 2$ if u_i is not adjacent to u_j and v_i is not adjacent to v_j. In order to satisfy this properties in this case, $|W_1| = \dim_{ls}(K_1 + G)$ while $|W_2| = \dim_{ls}(H)$ because $diam(H) \leq 2$. So, $\dim_{ls}(G + H) = \dim_{ls}(K_1 + G) + \dim_{ls}(H)$ if $diam(G) > 2$ and $diam(H) \leq 2$.

Case 3. If $diam(G) \leq 2$ and $diam(H) > 2$

According to Lemma 3.2, $d(u_i, u_j) = d(v_i, v_j) = 2$ if u_i is not adjacent to u_j and v_i is not adjacent to v_j. In order to satisfy this properties in this case, $|W_1| = \dim_{ls}(G)$ because $diam(G) \leq 2$ while $|W_2| = \dim_{ls}(K_1 + H)$. So, $\dim_{ls}(G + H) = \dim_{ls}(G) + \dim_{ls}(K_1 + H)$ if $diam(G) \leq 2$ and $diam(H) > 2$.

Case 4. If $diam(G) \leq 2$ and $diam(H) \leq 2$ or $\dim_{ls}(G) = \dim_{ls}(H) = 1$

Since $diam(G) \leq 2$ and $diam(H) \leq 2$ then $|W_1| = \dim_{ls}(G)$ and $|W_2| = \dim_{ls}(H)$. Assume $\dim_{ls}(G + H) = \dim_{ls}(G) + \dim_{ls}(H)$. Let $u_k \in W_1$ and $v_k \in W_2$, then there is a pair of adjacent vertice (u_i, v_j) where $u_i \notin W_1, v_j \notin W_2$, u_i adjacent with u_k, and v_j adjacent with v_k that applies $u_i \notin I[v_j, u_k], u_i \notin I[v_j, v_k], v_j \notin I[u_i, u_k]$, and $v_j \notin I[u_i, v_k]$. So, $\dim_{ls}(G + H) \neq \dim_{ls}(G) + \dim_{ls}(H)$. Following Lemma 3.1, $\dim_{ls}(G + H) = \dim_{ls}(G) + \dim_{ls}(H) + 1$ if $diam(G) \leq 2$ and $diam(H) \leq 2$ or $\dim_{ls}(G) = \dim_{ls}(H) = 1$. ■

Using theorems in Section 1, 2, and 3 we can get the local strong metric dimension of $G+H$ if G and H are two of the following graphs, which are path (P_n), cycle (C_n), complete graph (K_n), and star graph (S_n). Since $diam(P_n) \geq 2, diam(C_n) \geq 2$ for $n > 5$ and $diam(K_n) = 1 \leq 2, diam(S_n) = 2$ then according to Theorem 3.1:

i. $\dim_{ls}(P_n + P_m) = \dim_{ls}(K_1 + P_n) + \dim_{ls}(K_1 + P_m)$ if $n, m > 5$
ii. $\dim_{ls}(P_n + C_m) = \dim_{ls}(K_1 + P_n) + \dim_{ls}(K_1 + C_m)$ if $n, m > 5$
iii. $\dim_{ls}(P_n + K_m) = \dim_{ls}(K_1 + P_n) + \dim_{ls}(K_m)$
iv. $\dim_{ls}(P_n + S_m) = \dim_{ls}(K_1 + P_n) + \dim_{ls}(S_m)$
v. $\dim_{ls}(C_n + C_m) = \dim_{ls}(K_1 + C_n) + \dim_{ls}(K_1 + C_m)$ if $n, m > 5$
vi. $\dim_{ls}(C_n + K_m) = \dim_{ls}(K_1 + C_n) + \dim_{ls}(K_m)$
vii. $\dim_{ls}(C_n + S_m) = \dim_{ls}(K_1 + C_n) + \dim_{ls}(S_m)$
viii. $\dim_{ls}(K_n + K_m) = \dim_{ls}(K_n) + \dim_{ls}(K_m) + 1$
ix. $\dim_{ls}(K_n + S_m) = \dim_{ls}(K_n) + \dim_{ls}(S_m) + 1$
x. $\dim_{ls}(S_n + S_m) = \dim_{ls}(S_n) + \dim_{ls}(S_m) + 1$

4. Conclusion

In this paper we get the result regarding local strong metric dimension in the join of graphs $\boldsymbol{G+H}$ which is:

$$\dim_{ls}(G+H) = \begin{cases} \dim_{ls}(K_1+G) + \dim_{ls}(K_1+H) & \text{if } diam(G) > 2 \text{ and } diam(H) > 2 \\ \dim_{ls}(K_1+G) + \dim_{ls}(H) & \text{if } diam(G) > 2 \text{ and } diam(H) \leq 2 \\ \dim_{ls}(G) + \dim_{ls}(K_1+H) & \text{if } diam(G) \leq 2 \text{ and } diam(H) > 2 \\ \dim_{ls}(G) + \dim_{ls}(H) + 1 & \text{if } diam(G) \leq 2 \text{ and } diam(H) \leq 2 \\ & \text{or } \dim_{ls}(G) = \dim_{ls}(H) = 1 \end{cases}$$

In the next research, we suggest other researchers to develop the concept of local strong metric dimension into another concept of metric dimension, for example local strong complement metric dimension.

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